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TECHNIQUES OF DISTRIBUTIONS IN PERTURBATIVE QUANTUM FIELD THEORY.

(II) Applications to Theory of Multiloop Diagrams.

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ABSTRACT

The results of the mathematical theory of asymptotic operation developed in [1] are applied to problems of immediate physical interest. First, the problem of UV renormalization is analyzed from the viewpoint of asymptotic behaviour of integrands in momentum representation. A new prescription for UV renormalization in momentum space representation is presented (generalized minimal subtraction scheme); it ensures UV convergence of renormalized diagrams by construction, makes no use of special (e.g. dimensional) regularizations, and comprizes massless renormalization schemes (including the MS scheme). Then we present formal regularization-independent proofs of general formulae for Euclidean asymptotic expansions of renormalized Feynman diagrams (inlcuding short-distance OPE, heavy mass expansions and mixed asymptotic regimes etc.) derived earlier in the context of dimensional regularization [9], [10], [11]. This result, together with the new variant of UV renormalization, demonstrates the power of the new techniques based on a systematic use of the theory of distributions and establishes the method of As-operation as a comprehensive full-fledged—and inherently more powerful—alternative to the BPHZ approach.

1. Introduction

In the preceding paper [1] we undertook a regularization-independent formalization of the heuristic reasoning behind a series of publications [5], [6], [7], [8], [9], [10], [11] in which efficient methods of perturbative calculations were found (for references to various 2-, 3-, 4- and 5-loop calculations performed using that techniques see [12]). The new techniques for studying multiloop Feynman diagrams is based on a systematic use of the ideas of the distribution theory, and the key notion is that of asymptotic expansion in the sense of distributions [5], [9], [2]. A very general context in which to construct such expansions is established by the *extension principle* [5], [2]—an abstract functional-analytic proposition analogous to the classic Hahn-Banach theorem. A specific realization of the recipe implied by the extension principle—and the key instrument of our techniques—is the so-called asymptotic operation (As-operation) [9], [2]. The Euclidean version of As-operation constructed in [9], [2] is defined on a class of products of singular functions comprising integrands of Euclidean multiloop Feynman diagrams. The As-operation returns their expansions in powers and logarithms of a small parameter (e.g., a mass) in the sense of distributions.

In the first paper [1] an analytic technique was developed for describing singularities of distributions, as well as a combinatorial formalism (*universum of graphs*) to work with hierarchies of graphs and their subgraphs. Such a formalism makes it easy to utilize inherent recursive structures in problems involving multiloop diagrams. As a warm-up exercise, a very compact proof of a (localized) version of the familiar Bogoliubov-Parasiuk theorem in coordinate representation was presented with a purpose of illustrating in detail the typical ways of reasoning within the new techniques.

It was constructively proved that the asymptotic expansions in powers and logarithms of the small parameter exist for a large class of products of singular functions that includes integrands of Euclidean Feynman diagrams in momentum representation. The simplest example of such an expansion is with respect to a mass in a product of propagators, while the As-operation applied to such a product yields asymptotic expansions in powers and logs of the small mass, with coefficients given by compact explicit expressions suitable for studying expansion problems within the framework of applied Quantum Field Theory.

In the present paper we apply that general techniques to studying expansions of integrated Euclidean Feynman diagrams in momentum representation.

Similarly to the first part of the review [1], our purpose here is not to construct a complete perturbation theory or prove a short-distance expansion for a particular model in full detail but rather to demonstrate the general techniques of our formalism. Thus we do not attempt to write out, say, criteria for UV convergence of non-renormalized diagrams; such results are well-known and present little interest *per se*, and can be obtained in a straightforward manner from our general formulae whenever needed in a specific situation. Instead, we focus on how the techniques of the As-operation allows one to deal with the essential analytical aspects of such problems and to avoid combinatorial complexities of the usual approaches.

First of all, when performing integrations over infinite momentum space one encounters the well-known phenomenon of UV divergences that are due—in one of the possible interpretations—to slow decrease of Feynman integrands at infinite integration momenta. However, using the results of [1] one can explicitly extract those and only those terms in the asymptotic expansion of the integrand at infinite integration momenta, that are responsible for UV divergences. It turns out that such terms follow (with the opposite sign) the pattern of the terms to be introduced by the standard R-operation. Therefore, a direct subtraction of such terms from the integrand (with some natural precautions) results in a correct UV renormalization.

The UV finiteness of our construction is ensured *by definition* and what has to be proved is its equivalence to the standard formulation of the R-operation.[1] The class of subtraction schemes that naturally corresponds to the new definition of UV renormalization (the so-called *generalized minimal subtraction schemes*, or GMS schemes) comprises all massless schemes (including the MS scheme [17]) characterized by an extremely important property of polynomiality of the renormalization group functions in masses [20].

[1] Which, strictly speaking, is superfluous because one could prove correctness of the new representation directly, without reducing it to the standard construction. Such a proof would require results like existence of short-distance OPE (see e.g. [30]); cf. also [26]) which is also considered in the present paper.

A definition of the ultraviolet R-operation as a procedure of subtraction of asymptotics from the momentum representation integrands was first given by D. Slavnov et al. [18]. However, in [18] such subtractions are performed recursively with respect to loops, i.e. with respect to each integration momentum in turn, while in our approach asymptotics with respect to the entire collection of loop momenta are subtracted. This and the use of the As-operation for products of singular functions [1] in our approach allow one to easily exhibit the pattern of subtractions that is characteristic of the standard definition of the R-operation.

Our definition of the R-operation was first published in a complete form in [3]. It develops an idea from [9] and emerged from a study of diagrammatic interpretation of the non-trivial terms generated by the As-operation, not without an influence of [18].

An extremely important observation is that the coefficients of the As-operation constructed in [1] turn out to be exactly renormalized Feynman diagrams corresponding to subgraphs of the initial diagram. This fact has a dramatic technical impact on the problem of asymptotic expansions of Feynman diagrams, because it allows one to restore global OPE from expansions of individual diagrams in a very simple and straightforward fashion without any of the complexities of the combinatorial techniques of the BPHZ method.[2]

The second application of the theory of As-operation that we consider addresses the problem of Euclidean asymptotic expansions of renormalized multiloop diagrams. We present a compact and straightforward derivation of general Euclidean asymptotic expansions in the form of As-operation for integrated diagrams that was first introduced in [11]. Then the combinatorial techniques developed in [11] immediately allows one to obtain expansions for perturbative Green functions in OPE-like form.

The importance and feasibility of the general problem of Euclidean asymptotic expansions was realized in [9], [10], [11]. In those papers, a compact derivation of closed general formulae for such expansions was presented. The derivation of [9], [10], [11], however, aimed at obtaining the results in a shortest way and in a form immediately useful for phenomenological applications, so that a heavy use was made of the dimensional regularization and the MS scheme [17].[3] This left an open question of to what extent the results of [9], [10], [11] are independent of regularization.

Apart from the general interest, there are also very practical reasons for developing a regularization independent formalism. First of all, there are the notorious difficulties that the dimensional regularization encounters when applied to models involving γ_5 or supersymmetry. In particular, the γ_5 problem emerges if one wishes to use the chirality representation-based formalisms that are used to facilitate the enormously cumbersome gauge algebra in calculations of radiative corrections in QCD (cf. e.g. the superstring theory-inspired formalism developed in [32]).

Another reason is the breakdown of dimensional regularization in non-Euclidean asymptotic expansion problems [31], so that the regularization-independent formalism seems to be the only basis for construction of practical algorithms in the non-Euclidean case. This effect is connected with the insistence on expansions in "perfectly factorized" form, which is important for the following reasons:

It was an important realization of [7], [9], [10], [11] that a proof of any asymptotic expansion—be it Wilson's OPE or a heavy-mass expansion or asymptotics of the quark formfactor in the Sudakov regime—is phenomenologically irrelevant unless the result exhibits *perfect factorization* of large and small parameters. At the technical level of diagram-by-diagram expansions, perfect factorization means that the expansions run in pure powers and logarithms of the expansion parameter. Such expansions possess the property of uniqueness (cf. the discussion in [1], §15.4) which is tremendously useful from the technical point of view; for example, one immediately obtains that the As-operation commutes with multiplications by polynomials (see [1] and [10]). Another example is that one need not worry about properties like gauge invariance of the expansion in a given approximation: such properties are inherited by the expansion termwise from the initial amplitude, provided the expansion is

[2] It should be noted that the combinatorial constructions that naturally emerged within the framework of the theory of As-operation [11] (in particular, the inverse R-operation) caused an overhaul of how the combinatorial aspects are treated in the BPHZ theory (see [27])—even if the basic nature of the old approach (a complete resolution of all recursions) does not allow one to avoid having to deal with the rather cumbersome multiple summation formulas etc.

[3] One can ponder on the tremendous heuristic potential of the dimensional regularization and the MS scheme. Although the understanding of the analytic aspects of the problem—including existence of the representation of UV renormalization in the GMS form—was hardly lacking in [9], [10], [11], the compact presentation given there turned out feasible due to compactness of the formalism of dimensional regularization and its property to nullify certain types of scaleless integrals.

"perfect" in the above sense.[4]

For the above reasons, we consider it our major task to clarify the issue of existence of "perfect" expansions in regularization-independent way.

It is interesting that the derivation of OPE and, more generally, Euclidean asymptotic expansions presented in this paper—being more formalized than that of [9], [10], [11]—leads to a final formula which is much easier to deal with at the final stage of obtaining expansions for Green functions in a global "exponentiated" form. For example, unlike [11], we don't have to study inversion of the R-operation.[5]

However, as was stressed in [9], the derivation presented there was geared to the calculational needs of applied Quantum Field Theory (primarily, applications to perturbative Quantum Chromodynamics) and, therefore, dealt explicitly with UV counterterms etc. From practical point of view, the formalism of the present paper offers, at least in its current form, no advantages as compared with the explicit recipes of [9], [10], [11]—*provided* one can perform the calculations within dimensional regularization.

From theoretical point of view the formalization undertaken in the present series is more than just an exercise in rigour: there is the major unsolved problem of asymptotic expansions in non-Euclidean regimes, and it seems to be intrinsically intractable by the BPHZ method.[6] On the other hand, extension of the As-operation to non-Euclidean regimes—taking into account the accumulated experience [23] which only needs to be properly organized within an adequate technical framework—seems to be a matter of near future. We expect the experience gained in Euclidean problems to play a crucial role in the more complicated cases [33].

One of the main points in any proof of OPE is to study the interaction of UV renormalization and the expansion proper. As we are going to show, within our formalism the problem reduces to double asymptotic expansions in the sense of distributions. Indeed, a renormalized Feynman diagram in the GMS formulation has a form of an integral of the remainder of the As-expansion of the integrand in the regime when all the dimensional parameters of the diagram are much less than the implicit UV cutoff. When one applies the second expansion with respect to some of the diagram's masses or external momenta, there emerges, essentially, a double As-expansion. All one has to prove is that the double expansion thus obtained factorizes into a composition of two commuting As-expansions and that the remainder of such a double expansion is bounded by a factorizable function of the small parameters. This is done by a straightforward extension of the analytical techniques of [1].

The plan of the paper is as follows (as in [1], sections contain a preamble where further comments on its contents and results can be found).

In section 2 we present motivations and a definition of an operation $\mathcal{R}$ which subtracts asymptotics at large loop momenta from the integrand of a multiloop diagram, thus ensuring UV finiteness of the latter. In section 3 the explicit expressions for the As-operation from [1] are used to (partially) restrict the arbitrariness in the definition of the operation $\mathcal{R}$. For the purposes of illustration, section 4 establishes equivalence of $\mathcal{R}$ (with the arbitrary constants properly fixed) and the R-operation in the MS scheme.

In section 5 we obtain a useful representation for the operation $\mathcal{R}$ and apply it to deriving a "renormalization-group transformation" of the $\mathcal{R}$-renormalized diagrams. This transformation has exactly the form that is characteristic of the R-operation.

In section 6 the problem of the asymptotic expansion of renormalized Feynman diagrams is reviewed, its heuristic analysis from the point of view of As-operation is given, and a recursive expansion formula is obtained. Section 7 is devoted to combinatorial analysis of that formula. In section 8 a convenient expression for expanded renormalized diagrams is obtained in a combinatorial form similar to UV R-operation, and its exponentiation

[4] As was pointed out to us by J.C. Collins, such a property should be even more important for the problem of asymptotic expansions in Minkowskian regimes where both gauge invariance plays a greater role for phenomenological reasons, and the expansions one has to deal with are considerably more complicated. It may be said that the "relentless pursuit of perfection" (in the above sense) is one of the characteristic differences of the philosophy of the As-operation from the old BPHZ paradigm.

[5] Although we do use it in establishing—for the purposes of illustration—a connection of our GMS prescription to the standard MS scheme.

[6] Because the pattern of recursions involved is so much more complicated than in the Euclidean case that it does not seem possible to resolve them in an explicit manner [13].

on perturbative Green functions is considered. All analytic details are dealt with in section 9 where the key theorem on double asymptotic expansion is formulated and proved. Some less important technical results are relegated to two appendices.

For simplicity we will assume that the external momenta of the diagrams to be renormalized are fixed at non-exceptional values. This assumption, however, is inessential, and our results can be extended to the case of diagrams considered as distributions with respect to their external momenta.

The notations used in the present paper are the same as in [1].

2. Motivations and definition of the operation $\mathcal{R}$.

When studying the UV behaviour of momentum space Feynman integrands one normally invokes the Weinberg theorem [28] which supplies sufficient criteria for convergence of multidimensional integrals over infinite regions. That famous theorem, however, is very general and does not in the least take into account specific properties of Feynman diagrams. It turns out possible to effectively reduce the problem to one-dimensional integrals if one employs the techniques of the As-operation for products of singular functions developed in [1].

After fixing some notations in §2.1, in §2.2 we show how the As-operation emerges in the study of UV convergence of Feynman diagrams. In §2.3 we introduce a new definition for the procedure of elimination of UV divergences (the operation $\mathcal{R}$) which will be shown in subsequent sections to be equivalent to the standard R-operation.

§ 2.1. The momentum space integrand.

Let $G(p_G, \kappa_G)$ be the momentum-space integrand of an l-loop 1PI unrenormalized Feynman diagram, which depends on the set of D-dimensional (Euclidean) integration momenta $p_G = (p_1, \ldots p_l)$ (the aggregate variable p_G runs over the linear space P_G of $D \times l$ dimensions) as well as on some external parameters (masses and momenta) that are collectively denoted as κ_G. For simplicity the external momenta are also taken to be Euclidean but this limitation is inessential (cf. §19.3 of [1]).

When studying the expression $G(p_G, \kappa_G)$ it is convenient to use the formalism and notations developed in [1]. In particular, $G(p_G, \kappa_G)$ can be considered as a special case of the graph in the sense of [1] (cf. examples in §7.8 in [1]). Such special graphs (i.e. corresponding to integrands of Feynman diagrams in momentum representation) can be conveniently called *F-graphs*. To distinguish subgraphs in the sense of [1] from the 1PI subgraphs that are used in the theory of the R-operation, we will use the terms *IR-subgraphs* and *UV-subgraphs*, respectively.

The notations IR$[G]$ and UV$[G]$ will be used to denote the sets of all *IR*- and *UV*-subgraphs of a given F-graph G. (In the notations of [1], IR$[G] = S[G]$; we will use the term s-subgraph when discussing general properties of As-operation, while the term *IR*-subgraph is used when *UV*-subgraphs enter into consideration.)

We wish to consider convergence of the integral

$$\int dp_G \, G(p_G, \kappa_G) \tag{2.1}$$

at $p_G \to \infty$.

§ 2.2. Reduction to As-expansion problem.

Introduce into the integrand of (2.1) an arbitrary smooth cutoff function

$$\Phi(p_G/\Lambda) = \int_0^\Lambda d\lambda/\lambda \, \phi(p_G/\lambda),$$

where $\Phi(p_G) \in \mathcal{D}(P_G)$ and $\Phi(p_G) = 1$ in a neighbourhood of $p_G = 0$, and $\phi(p_G)$ differs from zero in a spherical layer. Then change the order of integrations over λ and p_G:

$$\int dp_G\, \Phi(p_G/\Lambda)G(p_G, \kappa_G) = \int_0^\Lambda d\lambda/\lambda \Big[\int dp_G\, \phi(p_G/\lambda)G(p_G, \kappa_G)\Big]. \tag{2.2}$$

We see that the behaviour of the l.h.s. at $\Lambda \to \infty$ is determined by the behaviour at $\lambda \to \infty$ of the square-bracketed expression on the r.h.s. which, after rescaling $p_G \to \lambda p_G$ and taking into account that $G(p_G, \kappa_G)$ has a definite dimension $\dim G$, takes the form:

$$\lambda^{\omega_G} \int dp_G\, \phi(p_G)G(p_G, \kappa_G/\lambda). \tag{2.3}$$

Here $\omega_G = D \times l - \dim G$ coincides with the usual index of UV divergence.

So, we have reduced the problem of studying asymptotics of $G(p_G, \kappa_G)$ at large momenta to studying the asymptotic expansion at $\lambda \to \infty$ of the integrals (2.3) with test functions $\phi(p_G)$ which are zero in neighbourhoods of p_G=0, in other words, to the problem of asymptotic expansion at $\lambda \to \infty$ of the expression $G(p_G, \kappa_G/\lambda)$ considered as a distribution over test functions from the space $\mathcal{D}(P_G\backslash\{0\})$. Up to a redefinition of the expansion parameter $\kappa_G/\lambda \to \kappa$, this is a special case of the more general problem which was formulated and explicitly solved in [1]. From [1] it follows that there exists (and can be presented in an explicit form—see eq.(20.16) in [1]) a unique asymptotic expansion of $G(p_G, \kappa_G/\lambda)$ in powers and logarithms of λ in the sense of the distribution theory:

$$G(p_G, \kappa_G/\lambda) \underset{\lambda\to\infty}{\simeq} \mathbf{As}'_{\kappa_G}\circ G(p_G, \kappa_G/\lambda) = \sum_{n\geq 0} \lambda^{d-n} \sum_i C_{n,i}(\ln\lambda)G_{n,i}(p_G). \tag{2.4}$$

Here $C_{n,i}(\ln\kappa)$ are polynomials of $\ln\kappa, G_{n,i}(p_G)$ are distributions on $\mathcal{D}(P_G\backslash\{0\})$, and d is a numeric constant. (The expansion which is valid on $\mathcal{D}(P_G)$ is denoted as $\mathbf{As}$. It is not used in the context of the problem of UV convergence.)

Each term in the sum over n in (2.4) inherits the dimensional properties of the l.h.s., so that a straightforward power counting allows one to determine exactly which terms on the r.h.s. are responsible for UV divergences at $\Lambda \to \infty$ in (2.2) (namely, the terms with $\omega_G + d - n \geq 0$). It should be stressed that *any* UV divergence on the l.h.s. of (2.2) (even the ones corresponding to divergences in subgraphs in the case of a negative index for G as a whole), will give rise in (2.4) to a term resulting in a divergence after integration at large λ in (2.2). Subtraction of such terms from $G(p_G, \kappa_G/\lambda)$ will ensure UV convergence of (2.2).

§ 2.3. Definition of the operation $\mathcal{R}$.

It is important to note that the term with $\omega_G + d - n = 0$ which is responsible in (2.4) for the logarithmic divergence at $\lambda \to \infty$ (or, equivalently, at $p_G \to \infty$ in the integral $\int dp_G G(p_G, \kappa_G)$), also has a logarithmic singularity at $\lambda \to 0$ (respectively, at $p_G \to 0$). This singularity is localized at an isolated point ($p_G = 0$) and can be removed (i.e. $\mathbf{As}'_{\kappa_G}\circ G(p_G, \kappa_G)$ can be extended to a distribution over $\mathcal{D}(P_G)$) with the help of the operators $\tilde{\mathbf{r}}$ that were introduced and studied in [1]. Such an extension is, of course, not unique, but different extensions differ only by a distribution localized at $p_G = 0$ (i.e. by a superposition of derivatives of the δ-function).

Let us define the operation $\mathcal{R}$ which acts on a Feynman diagram G as follows:

$$\mathcal{R}\circ G \overset{\text{def}}{=} \lim_{\Lambda\to\infty} \int dp_G\, \Phi(p_G/\Lambda)\big[G(p_G, \kappa_G) - \tilde{\mathbf{r}}_f\circ\mathbf{As}'_{\kappa_G}\circ G(p_G, \kappa_G)\big], \tag{2.5}$$

where the subscript f indicates that the subtraction operator $\tilde{\mathbf{r}}_f$ is chosen in the natural "factorized" form (see below).

Formally speaking, the definition (2.5) subtracts the entire asymptotic expansion, not only the terms that are responsible for UV divergences. But because of the so-called *minimality property* (see [1], §10.9), each of the extra terms (the ones with $\omega_G + d - n < 0$) is nullified by termwise integration over p for all sufficiently large

Λ. This remarkable fact allows one *not* to indicate explicitly how many terms are retained in the As-operation in (2.5)—and we will normally not do that.

It is asserted that with some natural limitations on $\tilde{\mathbf{r}}_f$ the operation $\mathcal{R}$ is equivalent to the standard R-operation while the remaining arbitrariness in the choice of $\tilde{\mathbf{r}}_f$ will correspond to the ordinary renormalization arbitrariness of the R-operation. But prior to defining $\tilde{\mathbf{r}}_f$ it is necessary to study the structure of the As-operation in more detail.

Remark. Strictly speaking, even in simple cases—e.g. if there are massless particles in the model—$G(p_G, \kappa_G)$ may contain non-integrable singularities at finite p. A typical and the most important example is a pair of massless propagators with the same momentum flowing through them and separated by a self-energy insertion. (Such singularities are suppressed after integrations over the loop momenta of the insertion.) In this case such singularities should preliminarily be removed using e.g. the operation $\tilde{\mathbf{R}}$ defined in [1] (concerning definition of $\mathbf{As}$ on such expression see §18.4 in [1]) or by introducing non-zero masses into the corresponding lines and putting them to zero in the final expression (2.5). Whether the final result will be affected by such a redefinition or not, depends on the model and on how the UV divergences have been subtracted. In the mentioned example with the pair of massless propagators, no problems arise if the insertion is zero at its zero external momentum both before and after the UV renormalization which means that the insertion is proportional to the squared momentum. This normally happens in gauge models with a gauge invariant renormalization. If, on the other hand, the insertion does not possess such a property, then the interactions generate a non-zero mass for the corresponding particle and the perturbation theory which starts with massless particles should be considered incorrect. One can imagine more complicated situations, but a general analysis of such complications—which is more or less straightforward, but requires taking into account specific detailes of the particular problem—goes beyond the scope of the present paper. In what follows we will limit ourselves to the case when $G(p_G, \kappa_G)$ is locally integrable for all finite p_G—this happens e.g. when G does not contain massless propagators.

3. Definition of $\tilde{\mathbf{r}}_f$ and the structure of $\tilde{\mathbf{r}}_f\circ\mathbf{As}'_{\kappa_G}\circ G$.

In this section we complete the definition of the operation $\mathcal{R}$, eq.(2.5), by fixing $\tilde{\mathbf{r}}_f$, and study the structure of the resulting expressions using specific properties of Feynman diagrams. The technical results of this section form a basis for a detailed study of the properties of the subtraction procedure $\mathcal{R}$ in subsequent sections.

§ 3.1. Counterterms of As-operation and the operation $\mathcal{R}$.

Recall (§17.5 in [1]) that the As-operation is constructed using the special operation $\tilde{\mathbf{R}}$ with specially chosen finite counterterms depending on the expansion parameter. For the operation $\mathbf{As}'_\kappa$, one has the expression similar to eq.(20.16) of [1] but with $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{r}}$ replaced by $\tilde{\mathbf{R}}'$ and $\tilde{\mathbf{r}}'$, with summation over γ restricted to $\gamma < G$. Application of $\tilde{\mathbf{r}}_f$ is then equivalent to removing primes on the r.h.s.:

$$\tilde{\mathbf{r}}_f\circ\mathbf{As}'_\kappa\circ G = \sum_{\gamma<G} \tilde{\mathbf{R}}_f\circ\big[(\tilde{\mathbf{E}}_{f,\kappa}\circ\gamma)(\mathbf{T}_\kappa\circ G\backslash\gamma)\big], \tag{3.1}$$

where

$$\tilde{\mathbf{E}}_{f,\kappa}\circ\gamma = \sum_\beta \tilde{E}_{f,\gamma,\beta}(\kappa)\,\delta^\beta_\gamma. \tag{3.2}$$

The explicit expression for $\tilde{E}$ is given by (20.5) in [1] which, taking into account the definition of $\mathcal{R}$, can be rewritten as

$$\tilde{E}_{f,\gamma,\beta}(\kappa) \equiv \mathcal{R}\circ\big[\mathcal{P}^\beta_\gamma * \gamma\big] \equiv \mathcal{R}\circ\Big[\int dp_\gamma\, \mathcal{P}^\beta_\gamma(p_\gamma)\gamma(p_\gamma,\kappa)\Big]. \tag{3.3}$$

It is sufficient to notice that the multiplication by a polynomial commutes with $\mathbf{As}'$ owing to the uniqueness of the latter (§15.7 in [1]), and with $\tilde{\mathbf{r}}_f$ up to a variation of the finite counterterms.

§ 3.2. UV-subgraphs as co-subgraphs.

Now, return into the context of (2.5). Then G is an F-graph and eqs.(3.1)–(3.3) remain valid with κ replaced by κ_G.

We begin with a combinatorial observation. The summation in (3.1) runs over all s-subgraphs γ (which in the present case are called *IR*-subgraphs) of the F-graph G, $\gamma \neq G$. An *IR*-subgraph γ is an arbitrary set of lines and vertices of G which satisfies the completeness condition (see §7.2 in [1]). In the present case this means the following: (i) when all the external momenta are nullified as well as all the momenta flowing through the lines of γ, no other line of G will have its momentum nullified owing to the momentum conservation at vertices; (ii) γ contains all those and only those vertices of G (irrespective of whether or not there are non-zero external momenta entering into them) whose all incident lines belong to γ (see the examples of *IR*-subgraphs in Fig.1 and Fig.3, where such vertices are the fat ones).

Consider the complement of γ in G, denoted as $G\backslash\gamma$ referred to as co-subgraph in §7.5 of [1]. The graphical image for $G\backslash\gamma$ is obtained by deleting the lines and vertices belonging to γ from the diagram G. It is not difficult to see (cf. the example in Fig.3) that the connected components of $G\backslash\gamma$ are precisely UV-subgraphs of G. (By an UV-subgraph we mean a subset of vertices of G together with some of the lines connecting them, and the UV-subgraph must be 1PI and possess at least one loop. The standard definition of the R-operation in the MS scheme [20] uses such subgraphs. The UV-subgraphs are defined to be non-intersecting if they have no common vertices.) Conversely, the complement of any set of pairwise non-intersecting UV-subgraphs $\{\xi_i\}$, $\xi_i \neq G$, is a correct *IR*-subgraph. Thus, there is a one-to-one correspondence between *IR*-subgraphs γ of the F-graph G and sets of its non-intersecting proper UV-subgraphs, which is expressed by the following relation:

$$G\backslash\gamma = \prod_i \xi_i. \tag{3.4}$$

Hence, there are natural factorization structure in graphs that emerge in the context of perturbative quantum field theory. This simple fact is a key to establishing equivalence of the operation (2.5) with $\tilde{\mathbf{r}}_f$ defined in (3.1) to the standard R-operation.

Denote the set of the loop momenta of the UV-subgraph ξ_i as p_{ξ_i}. Then the variable p_G is split as follows (cf. Fig.3):

$$p_G = (p_\gamma, p_{\xi_1}, \ldots, p_{\xi_i}, \ldots), \tag{3.5}$$

where p_γ are the proper variables of the *IR*-subgraph γ (see §6.1 in [1]). Then

$$G(p_G, \kappa_G) \equiv \gamma(p_\gamma, \kappa_G) \prod_i \xi_i(p_{\xi_i}, p_\gamma, \kappa_G). \tag{3.6}$$

It remains to notice that if κ_{ξ_i} is the set of all external parameters of the UV-subgraph ξ_i (i.e. masses in its lines, and the momenta that are external to ξ_i irrespective of whether they are external or internal for $G \supset \xi_i$), then

$$\kappa_{\xi_i} = (p_\gamma, \kappa_G).$$

It is also convenient to introduce the notation

$$p_{G\backslash\gamma} \stackrel{\text{def}}{=} (p_{\xi_1}, \ldots, p_{\xi_i}, \ldots), \qquad \kappa_{G\backslash\gamma} \stackrel{\text{def}}{=} (p_\gamma, \kappa_G), \tag{3.7}$$

so that

$$G(p_G, \kappa_G) \equiv \gamma(p_\gamma, \kappa_G) \times G\backslash\gamma(p_{G\backslash\gamma}, \kappa_{G\backslash\gamma}). \tag{3.8}$$

§ 3.3. Construction of $\tilde{\mathbf{r}}_f \circ \mathbf{As} \circ G$.

Let us specify the constructions of §3.1 to the present case. First consider the operation $\tilde{\mathbf{R}}$.

Consider the collection of all 1PI Feynman diagrams. If G belongs to this collection then we denote its momentum space integrand as $G(p_G,\kappa_G)$, where p_G is the set of its loop momenta and κ_G denotes the set of its external parameters, i.e. masses and external momenta. $G(p_G,\kappa_G)$ is an F-graph in the sense of §2.1. All the UV-subgraphs of G also belong to the same collection.

Consider expressions of the form $\left[\mathcal{D}^\alpha_{\kappa_G} G(p_G,\kappa_G)\right]_{\kappa_G=0}$ where $\mathcal{D}$ are partial derivatives and α is a multiindex. Such expressions can be transformed into well-defined distributions over p_G using special operations $\tilde{\mathbf{R}}$ that were introduced in [1].

Assume that for each G from our collection such $\tilde{\mathbf{R}}=\tilde{\mathbf{R}}_{(G)}$ has somehow been fixed (we are using the same conventions as in [1] concerning the subscripts in brackets denoting (sub)graphs—cf. §4.3 in [1]). In particular, for each UV-subgraph $\xi(p_\xi,\kappa_\xi)=\xi(p_\xi,p_{G\backslash\xi},\kappa_G)$ from G the expressions $\tilde{\mathbf{R}}_{(\xi)}\circ\mathcal{D}^\alpha_{\kappa_\xi}\circ\xi\big|_{\kappa_\xi=0}$ are well-defined distributions over p_ξ. We need not—and *will not*—assume that there is any connection between $\tilde{\mathbf{R}}_{(G)}$ and $\tilde{\mathbf{R}}_{(\xi)}$, $\xi\in\mathrm{UV}[G]$, except for the special case when G is factorizable and ξ is one of its factors (cf. the definition $\tilde{\mathbf{R}}$ on factorizable graphs in §11.2 of [1]). In the latter case, the subgraph ξ is automatically an IR-subgraph (cf. §7.6 of [1]).

If $\mathbf{T}_{\kappa_G}$ is the Taylor expansion in κ_G then the expression $\tilde{\mathbf{R}}\circ\mathbf{T}_{\kappa_G}\circ G(p_G,\kappa_G)$ implies termwise application of $\tilde{\mathbf{R}}=\tilde{\mathbf{R}}_{(G)}$ which by definition commutes with powers of κ_G.

In order to use the results of §3.1, an operation $\tilde{\mathbf{R}}=\tilde{\mathbf{R}}_{(G\backslash\gamma)}$ should be defined for $G\backslash\gamma$ for each IR-subgraph γ. We may assume that $\tilde{\mathbf{R}}$ always satisfies the factorization condition of §11.2 in [1], so that if $G\backslash\gamma$ is factorized into several UV-subgraphs as in (3.8) then

$$\tilde{\mathbf{R}}_{(G\backslash\gamma)}\circ\mathbf{T}_{\kappa_{G\backslash\gamma}}\circ G\backslash\gamma(p_{G\backslash\gamma},\kappa_{G\backslash\gamma})=\prod_i\tilde{\mathbf{R}}_{(\xi_i)}\circ\mathbf{T}_{\kappa_i}\circ\xi_i(p_{\xi_i},\kappa_{\xi_i}),\tag{3.9}$$

where $\mathbf{T}_{\kappa_i}\equiv\mathbf{T}_{\kappa_{\xi_i}}$ and we have used the fact that the Taylor expansion also factorizes.

Now, instead of (14.5) of [1] one has:

$$\tilde{\mathbf{R}}_f\circ\left[\delta^\alpha_\gamma\mathbf{T}_{\kappa_G}\circ G\backslash\gamma\right]\stackrel{\text{def}}{=}\sum_{\alpha=\beta+\sum_i\beta_i}\delta^\beta_\gamma\prod_i(\beta_i!)^{-1}\tilde{\mathbf{R}}_{(\xi_i)}\circ\left(\mathcal{D}^{\beta_i}_\gamma\mathbf{T}_{\kappa_G}\circ\xi_i\right)_{p_\gamma=0},\tag{3.10}$$

and instead of (3.1):

$$\tilde{\mathbf{r}}_f\circ\mathbf{As}'_{\kappa_G}\circ G\stackrel{\text{def}}{=}\sum_{\gamma<G}\tilde{\mathbf{R}}_f\circ\Big[(\tilde{\mathbf{E}}_{\kappa_G}\circ\gamma)\prod_i\mathbf{T}_{\kappa_G}\circ\xi_i\Big],\tag{3.11}$$

where $\tilde{\mathbf{E}}_{\kappa_G}$ is given by (3.2) and (3.3) with $\kappa\to\kappa_G$.

It should be remembered that (3.11) is a definition of the operator $\tilde{\mathbf{r}}_f$ which removes the non-integrable singularity at $p_G=0$ of $\mathbf{As}'_{\kappa_G}\circ G$. The latter expression is itself defined uniquely, i.e. it is independent of how the intermediate renormalization is performed, which means that it is independent of $\tilde{\mathbf{R}}'_{f(G)}$.

The above definition for $\tilde{\mathbf{r}}_f$ completes the definition of the operation $\mathcal{R}$ in (2.5).

§ 3.4. "Factoring out" δ-functions.

Let us derive, using the above results, a useful representation for the expression (3.11). The point is that the factors $\tilde{\mathbf{E}}_{\kappa_G}\circ\gamma$ contain derivatives of δ-functions $\delta^\beta_\gamma(p_\gamma)$ (cf. (3.2)), but the factors ξ_i in (3.11) may depend on the arguments of such δ-functions. Integration of the δ-functions will cause ξ_i to be differentiated in p_γ. Therefore, where the factors ξ_i in (3.11) are differentiated only in κ_G, they will be—after integrating out the δ-functions—differentiated in κ_G and some of the components of p_γ. The following calculation makes this effect explicit.

Taking into account (3.2) and (3.3), one obtains the following expression for the r.h.s. of (3.11):

$$\sum_{\gamma<G}\sum_{\alpha}\sum_{\alpha=\beta+\Sigma_i\beta_i}\mathcal{R}\big[\mathcal{P}^{\alpha}_{\gamma}*\gamma\big]\delta^{\beta}_{\gamma}\prod_i\tilde{\mathbf{R}}_f\circ\big(\mathcal{D}^{\beta_i}_{\gamma}\mathbf{T}_{\kappa_G}\circ\xi_i\big)_{p_\gamma=0}. \tag{3.12}$$

Note that the proper variables p_γ of the *IR*-subgraph γ are divided into two groups of components (cf. Fig.3): $p_\gamma=(p^{\rm int}_\gamma,p^{\rm ext}_\gamma)$, where $p^{\rm int}_\gamma$ that are the loop momenta corresponding to the loops formed by the lines of γ, and $p^{\rm ext}_\gamma$ that are external for the *UV*-subgraphs ξ_i. (Note that, $p^{\rm ext}_\gamma$ are the momenta that flow through the "loose" external lines of γ; cf. Fig.3.) Furthermore, from among the components $p^{\rm ext}_\gamma$, one can select the momenta $p^{\rm ext}_i$ that are external with respect to ξ_i, so that the differentiations in the expression under $\tilde{\mathbf{R}}_f$ can be performed with respect to $p^{\rm ext}_i$.

The following factorization holds:

$$\mathcal{P}^{\alpha}_{\gamma}\equiv p^{\alpha}_{\gamma}=p^{\beta}_{\gamma}\times\prod_i[p^{\rm ext}_i]^{\beta_i}. \tag{3.13}$$

It is easy to see that for any $\varphi(p)$ the expression of the form $\sum_\beta p^\beta(\mathcal{D}^\beta\varphi(p))_{p=0}$ is simply the Taylor expansion in p.

Therefore, in (3.12) with (3.13) there will emerge, instead of the Taylor expansion of the *UV*-subgraphs ξ_i in κ_G, i.e. in the masses and momenta that are external for G, the Taylor expansion in κ_{ξ_i}, i.e. the masses and momenta that are external for ξ_i. It is important that the expansion in κ_{ξ_i} emerges irrespective of whether or not a momentum that is external for ξ_i is also external for the entire G.

Finally, using (3.12)–(3.13) we can rewrite (3.11) as follows:

$$\begin{aligned}\tilde{\mathbf{r}}_f\circ\mathbf{As}'_{\kappa_G}\circ G=\tilde{\mathbf{R}}\circ\mathbf{T}_{\kappa_G}\circ G&+\sum_{\emptyset<\gamma<G}\delta_\gamma\mathcal{R}\circ\Big[\Big(\prod_i W_i\Big)*\gamma\Big]\\&+\sum_{\emptyset<\gamma<G}\sum_{\beta>0}\delta^{\beta}_{\gamma}\mathcal{R}\circ\Big[\Big(p^{\beta}_{\gamma}\prod_i W_i\Big)*\gamma\Big],\end{aligned} \tag{3.14}$$

where:

$$W_i(p_{\xi_i},\kappa_{\xi_i})\overset{\rm def}{=}\tilde{\mathbf{R}}_{(\xi_i)}\circ\mathbf{T}_{\kappa_i}\circ\xi_i(p_{\xi_i},\kappa_{\xi_i}), \tag{3.15}$$

and the terms with $\gamma=\emptyset$ and $\beta=0$ in (3.14) have been separated from the rest of the sum.

§ 3.5. Remarks.

To conclude this section, a few comments concerning (3.14) are necessary.

(*i*) The terms in the last sum on the r.h.s. of (3.14) are inessential in the sense that they can be got rid of in $\mathcal{R}\circ G$ (see below §5.1).

(*ii*) Recall (see §3.2) that one could replace the summation over *IR*-subgraphs γ by a summation over all sets of pairwise non-intersecting proper *UV*-subgraphs $\{\xi_i\}$. Such a pattern of summation is typical of the standard representations of the *R*-operation.

(*iii*) Diagrammatically, the expressions

$$\mathcal{R}\circ\Big[\Big(\prod_i W_i\Big)*\gamma\Big] \tag{3.16}$$

are obtained by contracting the *UV*-subgraphs ξ_i in G to points and using $W_i(\kappa_{\xi_i},p_{\xi_i})$ for the corresponding vertex factors. The aggregates $W_i(\kappa_{\xi_i},p_{\xi_i})$ are "infinite-order" polynomials of $\kappa_{\xi_i}=(\kappa_G,p^{\rm ext}_i)$, the external parameters of ξ_i. Recall that the operation $*_\gamma$ involves only integrations over p_γ of which each $p^{\rm ext}_i$ is a

part, and does not affect the dependence on p_{ξ_i}, the loop momenta of ξ_i, with respect to which W_i are well-defined distributions. Such a replacement of UV-subgraphs by polynomials of momenta that are external to the subgraphs is again typical of the standard R-operation, but in the present case the result is "renormalized" using $\mathcal{R}$.

(iv) Note that when we will substitute (3.14) into (2.5), the integration over p_γ as a part of the integration over p_G in (2.5) will be killed by the δ-functions δ_γ^β in (3.14) and replaced by the integration over p_γ implied by the operation $*$.

(v) It should be pointed out that the coefficients $\tilde{E}_{f,\gamma,\beta}(\kappa)$ can be interpreted as the renormalized graph γ in which the connected components of $G\backslash\gamma$ are shrunk to points and some polynomials are inserted instead (cf. (3.3) and (3.13)).

So, we have fully defined the operation $\mathcal{R}$ and presented explicit expressions for it, which can serve, in particular, as a starting point for comparison of $\mathcal{R}$ with the R-operation in the MS scheme.

4. The operation $\mathcal{R}$ and the R-operation in the MS scheme.

In this section we will show that the class of renormalization schemes defined via the operation $\mathcal{R}$ contains the R-operation in the MS scheme. First in §4.1 we will formally introduce the dimensional regularization into the definition of $\mathcal{R}$ (2.5) and integrate out the δ-functions in (3.14). Then in §4.2 we will study the structure of the "counterterms" introduced by the operators M in (3.14). In particular, specific nullification properties of the dimensional regularization will be used to show that a suitable choice of the operations $\tilde{\mathbf{R}}$ results in the "counterterms" containing pure poles in $D-4$. Lastly, in §4.3 we show that (3.14) is exactly the representation of the R-operation in terms of the "subtraction operator" for the inverted R-operation, and then invoke the results of [11] where R^{-1} was studied and conclude that with the above choice for $\tilde{\mathbf{R}}$, $\mathcal{R}$ is the R-operation in the MS scheme.

§ 4.1. Operation $\mathcal{R}$ in dimensional regularization.

Let us introduce the dimensional regularization into (2.5) as follows:

$$\mathcal{R}\circ G \overset{\text{def}}{=} \lim_{\epsilon\to 0}\int dp_G^{[\epsilon]}\left[G(p_G,\kappa_G) - \tilde{\mathbf{r}}_f\circ\mathbf{As}'_{\kappa_G}\circ G(p_G,\kappa_G)\right], \tag{4.1}$$

where ϵ is the deviation of the complex parameter of the space-time dimension D from the canonical integer value, while all quantities—integration measures, δ-functions etc.—are considered to be D-dimensional.[7]

Since the coefficients (3.3) are finite by construction and enter as coefficients multiplying well-defined distributions, they can also be replaced by regularized versions. This means, in particular, that the "principal value" integration implied by the operation $*$ in (3.3) and, consequently, in (3.14), is replaced by a dimensionally-regularized integration.

Owing to (3.5) the integration measure factorizes:

$$dp_G^{[\epsilon]} = dp_\gamma^{[\epsilon]}\prod_i dp_{\xi_i}^{[\epsilon]}. \tag{4.2}$$

[7] We have in fact dimensionally regularized the convergent expression on the r.h.s. under the limit $\Lambda\to\infty$, commuted the limits with respect to ϵ and Λ and let $\Lambda\to\infty$ before taking off the dimensional regularization. Justification of such a procedure is beyond the scope of the present paper. Anyhow, there are all indications that such manipulations with convergent expressions cannot give rise to problems. Concerning the definition of dimensional regularization directly for integrals over momentum space see [29]. Note that a direct formalization of the original definition of [16] is rather straightforward, and will hopefully be considered in a future publication.

Then, substituting (3.14) into (2.5), one sees that the terms with $\beta \neq 0$ are killed and the result can be represented as follows:

$$\mathcal{R}\circ G^\epsilon = G^\epsilon - \sum_{\{\xi_i\}} \mathcal{R}\circ\Big(\prod_i M^\epsilon_{\xi_i}\circ G^\epsilon\Big). \tag{4.3}$$

where summation runs over all sets $\{\xi_i\}$ of pairwise non-intersecting proper UV-subgraphs of G including the one consisting of only G itself. The other notations used are as follows:

$$G^\epsilon \stackrel{\text{def}}{=} \int dp_G^{[\epsilon]} G(p_G, \kappa_G); \tag{4.4}$$

the action of M^ϵ_ξ, where ξ is a proper UV-subgraph in G, on G consists in replacing ξ in G by a vertex to which there corresponds the following factor:

$$\int dp_\xi^{[\epsilon]}\, \tilde{\mathbf{R}}_{(\xi)}\circ\mathbf{T}_{\kappa_\xi}\circ\xi(p_\xi, \kappa_\xi); \tag{4.5}$$

the action of $\mathcal{R}$ on the diagrams with UV-subgraphs thus shrunk is defined in the same way as in the case of the entire G, and we have adopted the convention that $\mathcal{R}\circ 1 \equiv 1$ and $\mathcal{R}$ is insensitive to the numerical coefficients in the vertex factors (see the discussion of the structure of (4.5) in the next subsection) so that $\mathcal{R}\circ M^\epsilon_G\circ G = M^\epsilon_G\circ G$.

§ 4.2. Analytic structure of counterterms.

Consider the analytic structure of (4.5).

The operation $\mathbf{T}_{\kappa_\xi}$ Taylor-expands the integrand of the UV-subgraph ξ in κ_ξ, i.e. in all masses and momenta that are external with respect to ξ—i.e. in all dimensional parameters on which ξ depends. The coefficients multiplying the powers of masses and external momenta are functions of the loop momenta p_ξ only, and as is well-known, the integration over all p_ξ in infinite bounds within dimensional regularization nullifies such functions. However, in our case the operation $\tilde{\mathbf{R}}$ adds to $\mathbf{T}_{\kappa_\xi}\circ\xi(p_\xi, \ldots)$ some new terms that are obtained by replacing some groups of factors by counterterms proportional to δ-functions so as to make the resulting expression locally integrable at all finite p_ξ. It should be stressed that $\tilde{\mathbf{R}}$ does not affect—and is insensitive to—the powers of the masses and external momenta generated by $\mathbf{T}_{\kappa_\xi}$, so that the coefficients of such counterterms can be chosen to be independent of κ_ξ. One can see that such coefficients can be chosen to contain only pure singularities with respect to ϵ (which are well-known to consist of poles in ϵ; cf. [16]) with numeric coefficients (up to a general factor μ^ϵ to some integer power, where μ is the 't Hooft's unit of mass—which should be introduced to preserve dimensionality of expressions that are added to $\mathbf{T}_{\kappa_\xi}\circ\xi(p_\xi, \kappa_\xi)$).

If a counterterm introduced by $\tilde{\mathbf{R}}$ is such that not all integrations over p_ξ are killed by δ-functions then such term is nullified by the integration over all p_ξ due to reasons already discussed. So, there remain only terms that are proportional to $\delta(p_\xi)$ and its derivatives. But the terms with derivatives are also nullified by the integration over p_ξ with unit weight. Therefore, only terms proportional to $\delta(p_\xi)$ without derivatives survive. One can check by power counting that the power of the polynomials of masses and momenta which (polynomials) multiply such counterterms is exactly ω_ξ, the standard index of UV divergence of the UV-subgraph ξ.

Finally, one concludes that with the natural choice of the operation $\tilde{\mathbf{R}}$ explained above, the expression (4.5) takes the form

$$\text{eq.}(4.5) = Z\left(\tfrac{1}{\epsilon}\right)\cdot \mathcal{P}(\kappa_\xi)\cdot \mu^{\epsilon n}, \tag{4.6}$$

where Z is a polynomial of $1/\epsilon$ with numeric coefficients (the most general case of $\tilde{\mathbf{R}}_f$ only differs by finite contributions to Z), $\mathcal{P}$ is a polynomial of order ω_{ξ_i} of masses and external momenta of ξ_i, while the factor $\mu^{\epsilon n}$ (where n is proportional to the number of loops of ξ_i) is standard within the dimensional regularization and is introduced to preserve correct dimensionality of different terms in the sum (4.3).

§ 4.3. Inverted R-operation.

Thus, the operator M_ξ^ϵ in (4.3) replaces the UV-subgraph ξ by a vertex to which the factor of the form (4.6) corresponds. Such a form is characteristic of the counterterms of the R-operation in the MS scheme. However, the expression (4.3) is not a correct representation for the R-operation because of the presence of $\mathcal{R}$ which acts on the diagram with ξ's shrunk to points. Therefore, M_ξ^ϵ cannot be the counterterm operator of the operation $\mathcal{R}$ if the latter is to be an R-operation.

Nevertheless, rewrite (4.3) as follows:

$$\mathcal{R}\circ\Big\{G^\epsilon + \sum_{\{\xi_i\}}\Big(\prod_i M_{\xi_i}^\epsilon\circ G^\epsilon\Big)\Big\} = G^\epsilon. \tag{4.7}$$

It is clear that the braced expression on the l.h.s. describes the operation that is exactly the inverse for $\mathcal{R}$:

$$\mathcal{R}^{-1}\circ G^\epsilon = G^\epsilon + \sum_{\{\xi_i\}}\Big(\prod_i M_{\xi_i}^\epsilon\circ G\Big). \tag{4.8}$$

The arrangement of the operators M on the r.h.s. follows exactly the same combinatorial pattern as that of the counterterm operators in the standard representation of the R-operation.

The inversion of the R-operation was first introduced and studied in [11], where it was realized that this concept streamlines the study of combinatorial aspects of R- and As-operations. The explicit expressions for the operators M_ξ^ϵ in the representation (4.8) were derived in [11] in terms of the counterterm operators of the R-operation. Those expressions are polynomials with purely numerical coefficients, so that if the counterterms of the R-operation have the form (4.6) then the counterterms of R^{-1} has the same form and vice versa. We conclude that if $\tilde{\mathbf{R}}$ has been chosen as described after (4.5) then $\mathcal{R}$ is exactly the R-operation in the MS scheme, while (4.3) is its representation in terms of the counterterm operator for the inverse R-operation.

5. Structure of the operation $\mathcal{R}$.

Calculations similar to those in the preceding section—integrating out the δ-functions that are explicitly present in (3.14)—can be performed without reference to dimensional regularization. This is done below in §5.1. The resulting representation (5.1) for the operation $\mathcal{R}$, in which the UV finiteness—in contrast to (2.5)—is no longer obvious, is nevertheless more convenient for other purposes: indeed, it possesses an explicit recursion structure that can be effectively used in proofs. Thus, in §5.2–§5.5 the representation (5.1) is used to establish the fact that the arbitrariness in the definition of the operation $\mathcal{R}$ is equivalent to introducing finite counterterms for UV-subgraphs in exactly the same way as in the case of the standard R-operation. Concluding remarks concerning proofs of correctness of the operation $\mathcal{R}$ are contained in §5.6.

§ 5.1. Natural regularization of renormalized diagrams.

Let us substitute (3.14) into the definition (2.5) and integrate out the δ-functions δ_γ^β. Then the derivatives $\mathcal{D}_\gamma$ (cf. (14.1) in [1]) will switch over onto the cut-off function $\Phi_G \equiv \Phi$ which is arbitrary except that $\Phi_G \in \mathcal{D}(P_G)$. As was indicated in [1] the cutoffs can be chosen to satisfy restrictions (8.15), (8.16) of [1]. Assuming such choice of the cutoffs one obtains the following expression:

$$\mathcal{R}\circ G = \lim_{\Lambda\to\infty}\Big[G^\Lambda - \sum_{\{\xi_i\}}\mathcal{R}\circ\Big(\prod_i M_{\xi_i}^\Lambda\circ G\Big)\Big], \tag{5.1}$$

whose structure is completely analogous to that of (4.3) but

$$G^\Lambda \stackrel{\text{def}}{=} \int dp_G\, \Phi_G(p_G/\Lambda) G(p_G, \kappa_G), \tag{5.2}$$

and the vertex factors introduced by the operators M^Λ are as follows:

$$M_\xi^\Lambda \circ \xi = \int dp_\xi \, \Phi_\xi(p_\xi/\Lambda) \big[\tilde{\mathbf{R}} \circ \mathbf{T}_{\kappa_\xi} \circ \xi(p_\xi, \kappa_\xi)\big]. \tag{5.3}$$

It is obviously a series in κ_ξ, i.e. in masses and external momenta of ξ. However, owing to the minimality property of $\tilde{\mathbf{R}}$ (see §10.9 in [1]) all terms of a sufficiently high order in this series will vanish in the limit $\Lambda \to \infty$. In fact, as can be seen from sect.13 in [1], the vanishing terms become zero for all sufficiently large Λ and, therefore, can be omitted not only from (5.2) but also from (5.1). The non-vanishing terms constitute—as is easily seen by power counting—a polynomial in κ_ξ of exactly the order ω_ξ, the standard index of UV divergence.

§ 5.2. Finite renormalizations.

Let us study the arbitrariness in the definition of $\mathcal{R}$. (Recall that the operations $\tilde{\mathbf{R}}$ used in the construction—see e.g. (5.3)—are not defined uniquely.) We are going to demonstrate that this arbitrariness has exactly the same structure as in the case of the standard R-operations.

More precisely, let the indices a and b mark two sets of the operations $\tilde{\mathbf{R}}$ and, correspondingly, the two operations $\mathcal{R}$. Then, as we are going to demonstrate shortly,

$$\mathcal{R}_a \circ G = \mathcal{R}_b \circ G + \sum_{\{\xi_i\}} \mathcal{R}_b \circ \Big(\prod_i M_{\xi_i}^{ab} \circ G \Big), \tag{5.4}$$

which has the meaning similar to (5.1) except that the operator M_ξ^{ab} replaces the UV-subgraph ξ by a polynomial of its masses and external momenta of order ω_ξ, with finite coefficients. (Explicit expressions for M are given in (5.10).) For G without non-trivial UV-subgraphs (which corresponds to absence of IR-subgraphs $\gamma < G$) eq.(5.4) reduces to

$$\mathcal{R}_a \circ G = \mathcal{R}_b \circ G + m_G^{ab}, \tag{5.5}$$

which is obvious from the definitions.

In the most general case, eq.(5.4) is proved by induction using the representation (5.1). The reasoning can be made rather elementary and fully explicit with suitable notations introduced in the next subsection.

§ 5.3. Hierarchy of UV-subgraphs.

The representation (5.1) relates a diagram G with the diagrams obtained from G by shrinking some of the UV-subgraphs of G to points and inserting some factors into the new vertices that are polynomials in the external parameters of the corresponding UV-subgraphs. Therefore, an inductive proof should run over the entire set of all "secondary" diagrams thus obtained from G. Let us parameterize such secondary diagrams.

Let ξ be an aggregate index that consists of two labels: the first one, v_ξ, denotes an UV-subgraph from G, while the second one, α_ξ, indicates which polynomial of κ_ξ (the external parameters of the UV-subgraph v_ξ) is chosen for the vertex obtained by shrinking v_ξ to a point (α_ξ runs over a full linearly independent set of polynomials of κ_ξ).

Let $\{\xi\}$ be a set of such indices that corresponds to a set of pair-wise non-intersecting UV-subgraphs v_ξ, $\xi \in \{\xi\}$. Then $G\{\xi\}$ denotes the diagram obtained by shrinking the subgraphs v_ξ, $\xi \in \{\xi\}$, in G to points and inserting the α_ξ-th polynomials instead. A summation over $\{\xi\}$ implies both a summation over some collection of sets of pair-wise non-intersecting UV-subgraphs, and the corresponding summation over all the relevant labels α_ξ. A product over $\{\xi\}$ implies a product over only the sets of UV-subgraphs, with the corresponding labels α_ξ fixed.

It will also be convenient to denote the UV-subgraph v_ξ as "the UV-subgraph ξ", and $\{\xi\}$, "the set of UV-subgraphs".

Let ξ be an UV-subgraph from G, and let $\{\Xi\}$ be a set of UV-subgraphs such that $\Xi \subseteq \xi$ or $\Xi \cap \xi \neq \emptyset$ for all $\Xi \in \{\Xi\}$. Then $\xi\{\Xi\}$ is the UV-subgraph in $G\{\Xi\}$ that is obtained from ξ by shrinking to points those Ξ that lie within ξ. If a confusion is excluded, we will write just ξ instead of $\xi\{\Xi\}$.

Finally, $\{\xi\} > \{\Xi\}$ means that each ξ contains one or more Ξ's (the corresponding α_ξ and α_Ξ may be arbitrary). This relation specifies a partial ordering in the set of indices $\{\xi\}$ and the diagrams $G\{\xi\}$.

Using the above notations, eq.(5.1) describing the action of the operation $\mathcal{R}$ on a diagram $G\{\Xi\}$ can be represented as follows:

$$\mathcal{R}_a \circ G\{\Xi\} = \lim_{\Lambda\to\infty} \Big\{ G^\Lambda\{\Xi\} - \sum_{\{\xi\}>\{\Xi\}} \Big(\prod_\xi M^{\Lambda a}_{\xi\{\Xi\}} \Big) \mathcal{R}_a \circ G\{\xi\} \Big\}, \tag{5.6}$$

where all $M^{\Lambda a}_{\xi\{\Xi\}}$ are numerical coefficients. If $\{\Xi\} = \emptyset$ then (5.6) is the renormalized graph G.

Eq.(5.4) that we wish to prove now takes the form:

$$\mathcal{R}_a \circ G\{\Xi\} = \mathcal{R}_b \circ G\{\Xi\} + \sum_{\{\xi\}>\{\Xi\}} \Big(\prod_\xi M^{ab}_{\xi\{\Xi\}} \Big) \mathcal{R}_b \circ G\{\xi\}. \tag{5.7}$$

The expression for $M^{ab}_{\xi\{\Xi\}}$ is given below in (5.10).

The scenario of our proof of (5.7) is to use (5.6) in order to reexpress $\mathcal{R}_a$ on $G\{\Xi\}$ in terms of $\mathcal{R}_a$ on $G\{\xi\}$ for $\{\xi\} > \{\Xi\}$. Then we will use the inductive assumption that (5.7) is valid on $G\{\xi\}$ and reexpress $\mathcal{R}_a$ via $\mathcal{R}_b$. After that we will express $M^{\Lambda b}_{\xi\{\Xi\}}$ in terms of $M^{\Lambda a}_{\xi\{\Xi\}}$ and complete the proof. In the next subsection we will study how $M^{\Lambda b}_{\xi\{\Xi\}}$ and $M^{\Lambda a}_{\xi\{\Xi\}}$ are related and then prove eq.(5.7) in §5.5.

§ 5.4. Interrelation of counterterms in different schemes.

$M^{\Lambda b}_{\xi\{\Xi\}}$ is represented as (cf. (5.3)):

$$M^{\Lambda b}_{\xi\{\Xi\}} = \int dp_{\xi\{\Xi\}} \Phi_{\xi\{\Xi\}}(p_{\xi\{\Xi\}}/\Lambda) \big[\tilde{\mathbf{R}}_b \circ \mathbf{t}_{\alpha_\xi} \circ \xi\{\Xi\}(p_\xi, \kappa_\xi) \big], \tag{5.8}$$

where $\mathbf{t}_{\alpha_\xi}$ is the differential operator with respect to κ_ξ that corresponds to the α_ξ-th polynomial in κ_ξ in the Taylor expansion of ξ over its external momenta.

Using (14.7) of [1], we can reexpress $\tilde{\mathbf{R}}_b$ in (5.8) in terms of $\tilde{\mathbf{R}}_a$. Then we can integrate out the resulting δ-functional contributions similarly to (5.1). Using the properties of the cutoffs, we obtain:

$$M^{\Lambda b}_{\xi\{\Xi\}} = M^{\Lambda a}_{\xi\{\Xi\}} + \sum_{\{\upsilon\}} Z^{ba}_{\xi\{\Xi\}\{\nu\}} \prod_\upsilon M^{\Lambda a}_{\upsilon\{\Xi\}}, \tag{5.9}$$

where the summation runs over all sets of UV-subgraphs of $\xi\{\Xi\}$ excluding the set consisting of ξ itself but including the empty set (the corresponding term in the sum is just $Z^{ba}_{\xi\{\Xi\}}$).

We will prove the statement (5.7) together with the following expression for the finite counterterms M^{ab}:

$$M^{ab}_{\xi\{\Xi\}} = Z^{ba}_{\xi\{\Xi\}}. \tag{5.10}$$

First, note that the coefficients of the transformation $\mathcal{R}_a \to \mathcal{R}_b$ are related to the coefficients of the transformation $\tilde{\mathbf{R}}_b \to \tilde{\mathbf{R}}_a$ which should be compared with the fact that in sect.4 the inverse R-operation emerged.

Second, eq.(5.10) explicitly relates the arbitrariness in the definition of $\mathcal{R}$ with the underlying arbitrariness in the choice of the intermediate operation $\tilde{\mathbf{R}}$.

§ 5.5. Proof of the formula (5.7).

Let us prove (5.7) together with (5.10), assuming that they have been proved for all $G\{\xi\}$ with $\{\xi\} > \{\Xi\}$. Consider the difference $[\mathcal{R}_a - \mathcal{R}_b]\circ G\{\Xi\}$ and use (5.6):

$$\begin{aligned}
&[\mathcal{R}_a - \mathcal{R}_b]\circ G\{\Xi\} \\
&\quad = \lim_{\Lambda\to\infty}\Big\{-\sum_{\{\xi\}>\{\Xi\}}\Big(\prod_{\xi} M^{\Lambda a}_{\xi\{\Xi\}}\Big)\mathcal{R}_a\circ G\{\xi\} + \sum_{\{\xi\}>\{\Xi\}}\Big(\prod_{\xi} M^{\Lambda b}_{\xi\{\Xi\}}\Big)\mathcal{R}_b\circ G\{\xi\}\Big\}.
\end{aligned} \tag{5.11}$$

Then express $\mathcal{R}_a$ on the r.h.s. using the inductive assumption:

$$\mathcal{R}_a\circ G\{\xi\} = \mathcal{R}_b\circ G\{\xi\} + \sum_{\{\zeta\}>\{\xi\}}\Big(\prod_{\zeta} M^{ab}_{\zeta\{\xi\}}\Big)\mathcal{R}_b\circ G\{\zeta\}. \tag{5.12}$$

Substituting this into the r.h.s. of (5.11) one gets:

$$\begin{aligned}
(5.11) = \lim_{\Lambda\to\infty}\Big\{&\sum_{\{\xi\}>\{\Xi\}}\Big[\prod_{\xi} M^{\Lambda b}_{\xi\{\Xi\}} - \prod_{\xi} M^{\Lambda a}_{\xi\{\Xi\}}\Big]\mathcal{R}_b\circ G\{\xi\} \\
&- \sum_{\{\xi\}>\{\Xi\}}\Big(\prod_{\xi} M^{\Lambda a}_{\xi\{\Xi\}}\Big)\sum_{\{\zeta\}>\{\xi\}}\Big(\prod_{\zeta} M^{ab}_{\zeta\{\xi\}}\Big)\mathcal{R}_b\circ G\{\zeta\}\Big\}.
\end{aligned} \tag{5.13}$$

Now, the way $M^{\Lambda b}_{\xi\{\Xi\}}$ and $M^{\Lambda a}_{\xi\{\Xi\}}$ enter into the r.h.s. indicates that there should be some cancellations possible. Let us express $M^{\Lambda b}_{\xi\{\Xi\}}$ in terms of $M^{\Lambda a}_{\xi\{\Xi\}}$ using (5.9).

Consider the expression in the square brackets on the r.h.s. of (5.13). Using (5.9) one obtains a sum of terms that can be obtained from $\prod_\xi M^{\Lambda a}_{\xi\{\Xi\}}$ by replacing in all possible ways the factors for some ξ by the corresponding sums from the r.h.s. of (5.9). One can see that such a sum can be expressed as a sum over all sets of UV-subgraphs $\{\zeta\} > \{\xi\}$, as follows:

$$\prod_{\xi} M^{\Lambda b}_{\xi\{\Xi\}} - \prod_{\xi} M^{\Lambda a}_{\xi\{\Xi\}} = \sum_{\{\zeta\}>\{\xi\}}\prod_{\xi} Z^{ba}_{\xi\{\Xi\}\{\nu\}}\prod_{\zeta} M^{\Lambda a}_{\zeta\{\xi\}}, \tag{5.14}$$

where $Z^{ba}_{\alpha_\xi\{\alpha_\zeta\}}$ depend only on α_ζ for $\zeta\subset\xi$. Substituting this result into (5.13), renaming $\zeta\leftrightarrow\xi$ and rearranging the terms, we get:

$$\text{eq.}(5.11) = \sum_{\{\xi\}>\{\Xi\}}\Big(\prod_{\xi} Z^{ba}_{\xi\{\Xi\}}\Big)\mathcal{R}_b\circ G\{\xi\}. \tag{5.15}$$

Thus, we have proved that the renormalization-group transformations on individual diagrams have exactly the same form (5.4) for our operation $\mathcal{R}$ as for the standard R-operation. Moreover, we have established an explicit connection between the variations of the operation $\tilde{\mathbf{R}}$ and the induced variations of $\mathcal{R}$. This connection is expressed by (5.10).

§ 5.6. Remarks.

We would like to conclude this section with the following remark. To prove that $\mathcal{R}$ is equivalent to some other subtraction scheme requires introduction of a regularization that is natural for that scheme and commuting the limit $\Lambda\to\infty$ and the limit of taking off that regularization, as was done in §4.1 for the case of the dimensional regularization. Such an approach is rather inelegant.

It would be more satisfactory not to reduce one subtraction scheme to another but rather to prove that the perturbation series generated by an R-operation is a formal perturbative solution to some non-perturbatively

formulated equations. Such equations may need to require a more detailed knowledge of the properties of renormalized diagrams; in fact, it would be sufficient [30] to derive a short distance operator product expansion. The latter is a special case of the more general Euclidean asymptotic expansions, which we are going to derive in momentum representation.

6. Expansions of renormalized diagrams.

§ 6.1. Formulation of the problem.

The feasibility and usefulness of considering asymptotic expansions of Feynman diagrams for general Euclidean asymptotic regimes was realized in [9], [10], [11]. The formulation of the Euclidean asymptotic expansion problem follows [10] except that we now use GMS schemes instead of dimensional regularization and the MS scheme (which is a special case of GMS schemes) and consider only the simplified version of the expansion problem without contact terms.

Asymptotic regime. Let G be an arbitrary Euclidean multiloop Feynman diagram. Let $G(p,\ldots)$ be its unrenormalized momentum-space integrand, where p collectively denotes its integration (loop) momenta while dots stand for other dimensional parameters on which G also depends. Those parameters include masses (which enter the propagators of $G(p,\ldots)$) and external momenta, and will be referred to as *external parameters* of the diagram G. We wish to construct asymptotic expansion for the diagram G in the asymptotic regime when some of the external parameters of G are much larger than others.

Denote the large (or "heavy") external parameters of G collectively as M, and small (or "light") as m. Formally speaking, As-expansions that we study require presence of a scalar parameter with respect to which to expand. As an abuse of notation, we will use the same symbol m to represent such a parameter which goes to zero and to which all the light parameters are proportional. Thus, the asymptotic regime we wish to consider is described as $m \to 0$ and $M = O(1)$.

It makes sense to assume that neither the set M nor m are empty.

Perfect factorization. An extremely important requirement on expansions at the level of an individual diagram is that they should ran in powers and logs of the expansion parameter. This is a conretization of the fundamental requirement of perfect factorization of expansions of Green functions, which means that the coefficient functions depending on large parameters should not contain non-analytic (logarithmic) dependences on light parameters. It has been realized that only expansion possessing the property of perfect factorization have phenomenological significance; in particular, only such expansions are useful in models with massless particles like QCD. Moreover, such expansions possess the property of uniqueness which turns out to be extremely useful; e.g. it simplifies study of gauge properties of expansions since they then inherit the gauge properties of non-expanded Green functions (a formalism for studying gauge properties within the formalism of the As-operation was developed in [24]). Phenomenological and technical aspects of this requirement are discussed in detail in [10]. Here we only note that in the papers [14] where the fact of power-and-log nature of expansions was verified at a formal level for several asymptotic regimes, no convenient explicit formulae for coefficient functions of expansions were presented, while the standard BPHZ derivation of OPE resulted in expansions in which coefficient functions that contained all the dependence on the large momentum also depended on the light masses in a non-trivial way. Short distance OPE possessing the property of perfect factorization was obtained in [7]. An important consequence was the discovery of efficient calculational formulae for coefficient function of OPE [7], [8].[8]

[8]For a (attempt of) verification of some of the dimensionally regularized results of the theory of As-operation within the framework of the (modified) BPHZ method see [27]. The combinatorial part of [27] is strongly influenced by the theory of As-operation [11] but never attains transparency of the latter. On the other hand, the only completely explicit treatment of the analytical aspects of the theory of Euclidean asymptotic expansions remains the one given in [2], [4], [1] and the present paper.

Renormalization. We assume that the diagram G (i.e. the integral of $G(p, \ldots)$ over p in infinite bounds) is renormalized using the GMS prescription. The GMS prescription comprises all the schemes that possess the property of polynomial dependence of the corresponding renormalization group functions on dimensional parameters.[9] This is an important assumption both technically and conceptually.

Conceptually, the asymptotic expansions in masses and momenta obtained within such schemes possess, as we will see, the above mentioned property of perfect factorization.[10]

Technically, the GMS prescription amounts to subtraction from the integrand of those and only those terms in the asymptotic expansion in the UV regime that generate UV divergences, while the necessary modification of the logarithmic terms at zero momentum (the operation $\tilde{\mathbf{r}}$) does not affect the dependences on the dimensional parameters. The net effect, as we will see, is to trivialize the problem of expansion of renormalized diagrams by reducing it to a study of double As-expansions (see §6.3 below).

Renormalization introduces an additional external parameter for G besides its masses and external momenta. Such parameter is usually denoted as μ. The dependence on μ is known explicitly: $\mathcal{R}{\circ}G$ is a polynomial in $\log \mu$. Therefore, it is of no practical consequence whether μ is considered as a heavy or light parameter. For definiteness, we will treat it as a heavy parameter.

External momenta as fixed parameters. The simplest version of the expansion problem emerges if one fixes external momenta at some values and then treats them on an equal footing with masses. This is what we will assume for now. It should be emphasized that in principle it is not necessary to fix the momenta at generic non-zero or otherwise non-exceptional values as long as the initial expression one wishes to expand is well defined. We will discuss this point in more detail below. Here we only wish to note that the precise conditions of when a diagram is well-defined depend on the details of the structure of specific Feynman diagrams in a specific model, and it is not our aim to discuss such conditions. The only important thing is that our technique is insensitive to such details.

A somewhat more complicated (and more general) version of the problem would be to consider the diagram as a distribution in the external momenta. Here one expects additional terms to appear in the expansion; such "contact" terms should be proportional to δ-functions of linear combinations of the external momenta (cf. below the discussion of IR singularities of the non-expanded diagrams). This case will be considered in a separate publication. Note that within the framework of dimensional regularization and the MS scheme explicit expressions for contact terms were obtained in [11].

Linear restrictions on momenta. Another aspect of the problem is how one divides the external momenta into heavy and light. The point is that certain sums of heavy external momenta should be allowed to be light, i.e. $O(m)$, in some physically meaningful situations. This amounts to imposing linear restrictions on the heavy external momenta. Such restrictions were analyzed in [11] where an important class of *natural restrictions* was identified. The natural restrictions can be described as follows. One divides the external lines of G through which heavy external momenta flow, into several "bunches". Then for each bunch one imposes a single restriction that the (algebraic) sum of the corresponding momenta is $O(m)$ (i.e. it is equal to a combination of the light momenta only). Otherwise the heavy momenta are assumed to take generic values. Note that there always is at least one such restriction due to overall momentum conservation: the sum of all heavy momenta must be equal to the sum of all light momenta. As a rule our analysis at the level of individual diagrams is quite general, but the interpretation of the results at the level of Green functions turns out to be more transparent for asymptotic regimes with natural restrictions.

Infrared divergences. It should be pointed out that Feynman integrands in models with massless particles possess singularities at finite values of p due to massless propagators. Such singularities can formally be even non-integrable but, nevertheless, spurious in the sense that they cancel out after performing integrations and

[9] The first and most important example is the MS scheme [17], for which the polynomiality property was established in [20].

[10] The fact that the property of perfect factorization is a necessary condition for existence of OPE in the MS scheme was observed in [19]. For non-GMS schemes perfectly factorized expansions have a more complicated form.

taking into account specific algebraic properties of the integrands like gauge invariance (cf. the Kinoshita-Poggio-Quinn theorem [21]), or they may require special treatment whose exact form is determined by additional considerations. Such considerations are "orthogonal" to the expansion problem proper (in the sense explained below) and their discussion goes beyond the scope of the present paper. However, the following remarks can be made here.

One can remove such singularities from the initial integrand using a version of the special subtraction operation $\tilde{\mathbf{R}}$ introduced in [1]. Then one can perform all the reasoning of the present paper taking into account that our technique is essentially insensitive to presence of such $\tilde{\mathbf{R}}$ in the integrand (see §18.4 in [1] and our §7.4).

Alternatively, one can regularize such singularities by introducing a mass, m_0, for the massless particles, and after the expansion in m_0 is done, to consider the limit $m_0 \to 0$. This would be technically equivalent to considering double expansion in the regime $m_0 \ll m \ll M$. Such a problem can be studied by straightforward application of the results of the present paper. Indeed, our main concern here is exactly the extension of the results on simple As-expansions to the case of double As-expansions (see below §6.3). Needless to say, further extension to three-fold expansions etc. is completely straightforward. The net effect is that the multiple As-expansions of the above type can be obtained by performing simple expansions sequentially, in any order. Thus, one can first expand in the regime when m_0 and m are much less than M, and after that, perform termwise expansion of the result for $m_0 \ll m$. How many terms in the expansion in m_0 one should retain, and what one should do with the singularities in m_0 , is to be decided from the specifics of the problem. This way to proceed is essentially equivalent to the first one based on the use of the operation $\tilde{\mathbf{R}}$, because the As-expansion in m_0—as any As-expansion—can be expressed as an $\tilde{\mathbf{R}}$ with suitably chosen finite counterterms. However, introduction of a non-zero mass m_0 has an advantage of not requiring new notations.

Either way, the factors in the final expansion that depend on the heavy parameters of the problem (e.g. the coefficient functions of the OPE which depend on the heavy momentum Q) are independent of the IR structure of the initial diagram.[11] Only the factors in which the dependence on the light parameters is concentrated are sensitive to IR structure. This is essentially the property of perfect factorization.

To reiterate: what happens when m_0 is taken to zero depends on details like gauge invariance of the model and UV renormalization procedure adopted, and affects only those factors in the expansion which contain the non-trivial dependence on the light parameters. On the other hand, what we wish to focus on is the analytical aspects of the expansion problem proper, avoiding inessential details. It is sufficient to say in this respect that our techniques offers efficient ways to deal with such singularities. For the above reasons, in what follows we will simply ignore such singularities, in order to avoid unnecessary notational complications. This will allow us to concentrate on the non-trivial aspects of our techniques.

Lastly, it is convenient to assume that the set of heavy parameters M is non-empty, because otherwise the expansions in κ and m coincide and the problem degenerates into a trivial one. (Triviality means that the renormalized diagram itself is a power of m times a polynomial in $\log m$, so that its As-expansion coincides with the diagram itself.)

§ 6.2. Expansion of UV-regularized diagram.

A GMS renormalized diagram can be represented as follows (cf. (2.5)):

$$\mathcal{R}\circ G(M,m) \equiv \mathcal{R}\circ \int dp\, G(p,M,m) \tag{6.1}$$

$$\equiv \lim_{\Lambda\to\infty} \int dp\, \Phi^{\Lambda}(p) \big[G(p,M,m) - \tilde{\mathbf{r}}\circ \mathbf{As}'_{\kappa}\circ G(p,\kappa M,\kappa m) \big]_{\kappa=1}.$$

We have introduced an auxiliary parameter κ on the r.h.s. of (6.1) in order to formally describe the asymptotic expansion of the integrand in the UV regime in which all dimensional parameters of G are small as compared to the cutoff Λ.

[11] Another way to put it is to say that the OPE coefficient functions are analytic in the light mass parameters, including the regulator mass m_0. Therefore, whatever one does with m_0 afterwards (e.g. taking it to 0) will not, essentially, affect the coefficient functions.

What we ultimately wish to do is to determine the explicit expression for the As-expansion of (6.1) in $m \ll M$ which we denote by the same symbol **As** as we use for the As-operation on non-integrated products:

$$\mathcal{R}\circ G(M,m) \underset{m\to 0}{\simeq} \mathbf{As}_m\circ\mathcal{R}\circ G(M,m) = ? \tag{6.2}$$

Such a notation is very natural both in view of the general definitions of section 15 of [1] and because the new version of the operation **As** (6.2), as we will see, is closely related to the operation **As** already defined on non-integrated products. Note that the term "As-operation" was first introduced for *integrated* diagrams [11].

It is natural to try to find the expression for (6.2) by applying the operation $\mathbf{As}_m$, which has already been defined on products of singular functions, to the expression in square brackets on the r.h.s. of (6.1). As we will see, this is indeed possible. Moreover, the corresponding calculation exhibits a recursive pattern: in order to derive (6.2) for G itself one has to assume validity of an expansion of the type (6.2) for integrated renormalized graphs with a lesser number of loops than G.

Studying eq.(6.1) at fixed Λ. First of all, let us fix $\Lambda < \infty$ and study the expansion at $m \to 0$ of the resulting "regularized" integral. One can immediately write down the expansion for the first term on the r.h.s. of (6.1), $\int dp\, \Phi^\Lambda(p)\, G(p,M,m)$, by directly using the techniques of [1], i.e. by applying the As-operation to $G(p,M,m)$:

$$\int dp\, \Phi^\Lambda(p)\, G(p,M,m) \underset{m\to 0}{\simeq} \int dp\, \Phi^\Lambda(p)\, \mathbf{As}_m\circ G(p,M,m).$$

In order to expand the remaining contributions to (6.1), one should consider the structure of the expression $\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G$ in more detail. From eqs.(3.1)–(3.3) it follows that:

$$\begin{aligned} &\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,\kappa M,\kappa m)\big|_{\kappa=1} \\ &= \sum_{\gamma<G}\sum_\alpha \tilde{\mathbf{R}}\circ\big\{\delta_\alpha(p_\gamma)\,\mathbf{T}_\kappa\circ G\backslash\gamma(p,\kappa M,\kappa m)\big\}_{\kappa=1} \times \mathcal{R}\circ\int dp'_\gamma\, \mathcal{P}_\alpha(p'_\gamma)\,\gamma(p'_\gamma,M,m). \end{aligned} \tag{6.3}$$

(Note that the summation here runs over subgraphs corresponding to the singularities of the formal expansion in κ.[12])

As a convenient abuse of notations we will often omit κ in expressions like the r.h.s. of (6.3) everywhere except in $\mathbf{T}_\kappa$:

$$\mathbf{T}_\kappa\circ G\backslash\gamma(p,\kappa M,\kappa m)_{\kappa=1} \to \mathbf{T}_\kappa\circ G\backslash\gamma(p,M,m). \tag{6.4}$$

This should cause no ambiguity.

The r.h.s. of (6.3) is a sum of terms in which the dependences on the integration momenta p and on the external parameters are completely factorized. Indeed, the distribution $\tilde{\mathbf{R}}\circ\big\{\delta_\alpha(p_\gamma)\,\mathbf{T}_\kappa\circ G\backslash\gamma(p,\ldots)\big\}$ is a polynomial of the external parameters M and m (this is due to the action of $\mathbf{T}_\kappa$ and the fact that $\tilde{\mathbf{R}}$ does not affect M- and m-dependences). On the other hand, the factor $\mathcal{R}\circ\int dp'_\gamma\,\mathcal{P}_\alpha(p'_\gamma)\,\gamma(p'_\gamma,M,m)$ contains a non-trivial dependence on M and m but is independent of p. Moreover, the latter factor has the same form as the initial expression (6.1) up to a replacement $p \to p_\gamma$ and $G(p) \to \mathcal{P}_\alpha(p_\gamma)\times\gamma(p_\gamma)$. Therefore, it is natural to make an inductive assumption that the operation **As** has been defined on the products $\mathcal{R}\circ\int dp'_\gamma\,\mathcal{P}_\alpha(p'_\gamma)\,\gamma(p'_\gamma,M,m)$ for all $\gamma < G$. Then the As-expansion of (6.3) is given by applying such **As** to the last factor in (6.3):

$$\begin{aligned} \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,M,m) &\underset{m\to 0}{\simeq} \mathbf{As}_m\circ\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,M,m) \\ &= \sum_{\gamma<G}\sum_\alpha \tilde{\mathbf{R}}\circ\big\{\delta_\alpha(p_\gamma)\,\mathbf{T}_\kappa\circ G\backslash\gamma(p,M,m)\big\} \times \mathbf{As}_m\circ\mathcal{R}\circ\int dp'_\gamma\,\mathcal{P}_\alpha(p'_\gamma)\,\gamma(p'_\gamma,M,m). \end{aligned} \tag{6.5}$$

[12]We will also have to deal with singularities of, and the corresponding operation $\tilde{\mathbf{R}}^m$ for, the expansion in m. The resulting notational complications will be dealt with in §7.1 below.

It is important to note that there has emerged a compact recursive pattern which is characteristic of our techniques: expansion of the GMS renormalized diagram G (6.1) is reduced to an essentially similar problem but with a lesser number of integration momenta.

In the case of a single loop momentum in the initial diagram G (or one-dimensional p if one does not limit the discussion to Feynman diagrams proper), our expansion problem degenerates into a trivial one. Because of this and owing to the explicit recursive pattern in (6.5) we can assume that the problem has been solved for all GMS-renormalized diagrams with lesser number of loop momenta, i.e. that the operation $\mathbf{As}$ on the r.h.s. of (6.5) is well defined. Using this inductive assumption completes the expansion of the regularized integral.

Explicit expressions for $\mathbf{As}_m \circ \mathcal{R} \circ G$ which provide solution to the above recursive procedure, will be presented below in section 7.

§ 6.3. The limit $\Lambda \to \infty$.

The only important question of analytic nature that needs to be answered is whether the asymptotic expansion constructed above for fixed Λ remains such after taking the limit $\Lambda \to \infty$, i.e. whether the As-operation defined by the expression[13]:

$$\mathbf{As}_m \circ \mathcal{R} \circ \int dp\, G(p,M,m) \stackrel{\text{def}}{=} \lim_{\Lambda\to\infty} \int dp \circ \Phi^\Lambda(p) \mathbf{As}_m \circ \big[1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa\big] \circ G(p,M,m) \tag{6.6}$$

delivers a true As-expansion for the integral (6.1). The answer is yes, and it can be justified in two steps.

(i) As a first step, it is natural to consider existence of the limit on the r.h.s. of (6.6). To this end we split the integration region in (6.6) into two parts by introducing an intermediate cut-off at the radius μ in order to explicitly separate the non-trivial asymptotic region $p \to \infty$ from the point $p = 0$ where the expression is complicated by the operator $\tilde{\mathbf{r}}$ but the expansion is essentially straightforward:

$$\begin{aligned} \mathbf{As}_m \circ \mathcal{R} \circ \int dp\, G(p,M,m) &\equiv \int dp\, \Phi^\mu(p)\, \mathbf{As}_m \circ \big[1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa\big] \circ G(p,M,m) \\ &+ \lim_{\Lambda\to\infty} \int dp\, \Phi^\Lambda_\mu(p) \mathbf{As}_m \circ \big[1 - \mathbf{As}'_\kappa\big] \circ G(p,M,m). \end{aligned} \tag{6.7}$$

(For definition of the functions Φ see §8.5 of [1].) The second term in this expression will be finite if the two As-operations commute:

$$\mathbf{As}_m \circ \mathbf{As}'_\kappa = \mathbf{As}'_\kappa \circ \mathbf{As}_m. \tag{6.8}$$

Then the operator $1 - \mathbf{As}'_\kappa$ can be taken out to the left of $\mathbf{As}_m$, so that existence of the limit $\Lambda \to \infty$ will follow automatically (recall in this respect the motivations and construction of the operation $\mathcal{R}$ in section 2).

The commutativity (6.8) (see also (7.6)) is one of the central results of the present paper. Its nature is essentially algebraic. Here we only note that explicit formulae for $\mathbf{As} \circ \mathcal{R} \circ G$ follow from (7.6) (see section 7).

(ii) The second step is analytic in nature: one verifies that the remainder of the expansion (6.7) vanishes at the required rate as $m \to 0$. This is formally expressed as

$$\begin{aligned} &\big[1 - \mathbf{As}^n_m\big] \circ \mathcal{R} \circ \int dp\, G(p,M,m) \\ &\stackrel{\text{def}}{=} \lim_{\Lambda\to\infty} \int dp\, \Phi^\Lambda(p) \big[1 - \mathbf{As}^n_m\big] \circ \big[1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa\big] \circ G(p,M,m) = o(m^n). \end{aligned} \tag{6.9}$$

[13] As should be clear from the above construction, the composition of the two As-operations here is purely algebraic: the second operation is applied termwise to the series generated by the first one irrespective of approximation properties of the resulting expression.

To check (6.9), one splits the integration region as in (6.7):

$$\begin{aligned} \big[1 - \mathbf{As}_m^n\big] \circ \mathcal{R} \circ \int dp\, G(p, M, m) & \\ \equiv \int dp\, \Phi^\mu(p) \big[1 - \mathbf{As}_m^n\big] \circ \big[1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa\big] \circ G(p, M, m) & \\ + \lim_{\Lambda \to \infty} \int dp\, \Phi_\mu^\Lambda(p) \big[1 - \mathbf{As}_m^n\big] \circ \big[1 - \mathbf{As}'_\kappa\big] \circ G(p, M, m). & \end{aligned}$$

For the first term on the r.h.s. the estimate (6.9) is true by definition of $\mathbf{As}$ and because $\Phi^\mu(p)$ is an ordinary test function. For the second term, one represents $\Phi_\mu^\Lambda(p)$ as an integral over spherical layers of radius λ (see §8.5 of [1]):

$$\Phi_\mu^\Lambda(p) = \int_\mu^\Lambda \frac{d\lambda}{\lambda}\, \phi_\lambda(p). \tag{6.10}$$

Then one rescales the integration variable $p \to \lambda p$ and uses the uniformity properties of G to arrive at the following expression:

$$\int_\mu^\Lambda \frac{d\lambda}{\lambda} \int dp\, \phi_1(p)\, \big[1 - \mathbf{As}_m^n\big] \circ \big[1 - \mathbf{As}_{\kappa/\lambda}\big] \circ G(p, M\kappa/\lambda, m\kappa/\lambda). \tag{6.11}$$

Recall that one can retain only those terms in $\mathbf{As}'_\kappa$ that are responsible for UV divergences (see the text immediately after eq.(2.5)). Since λ always divides κ in the above expression, one can see that the $o(m^n)$ estimate for (6.11) follows from an estimate of the type

$$\left| \int dp\, \phi_\lambda(p)\, \big[1 - \mathbf{As}_m^n\big] \circ \big[1 - \mathbf{As}_\kappa^l\big] \circ G(p, \kappa M, \kappa m) \right| < o(m^n) \times o(\kappa^l). \tag{6.12}$$

This can be adopted as (part of) an exact analytic interpretation of the informal statement that the algebraic composition of the two (commuting) As-operations yields a true double asymptotic expansion in the sense of distributions for the integrand $G(p, \kappa M, \kappa m)$.

The inequality (6.12) is easy to understand at a heuristic level. Indeed, the remainder of an asymptotic expansion is often estimated (at least in the cases when the expansion has a relatively simple analytic nature—as in our case where one deals with expansions of integrals of rational functions, however cumbersome) by the last discarded term, which in our case is $O(m^{n+1}) \times O(\kappa^{l+1})$ (up to inessential logarithms). This immediately explains (6.12).

Actually, the inequalities that we prove in Theorem 1 (see below section 9) are more stringent than (6.12); in particular, they also describe dependence of the bounds on the support of the test function. This is needed in order to carry on the induction.

§ 6.4. Summary

We have exhibited the recursive structure of the As-operation for GMS renormalized diagrams and identified the inductive assumptions as well as the propositions that have to be proved. To proceed, we first have to study the structure of our expressions in more detail and derive an explicit expression for $\mathbf{As}_m \circ \mathcal{R} \circ \int dp\, G$. This will allow us to check the formal commutativity of the two As-operations $\mathbf{As}_m$ and $\mathbf{As}_\kappa$. Second, we have to present and prove the inequalities that guarantee validity of (6.12).

7. Explicit expressions for $\mathbf{As}_m \circ \mathcal{R} \circ G$.

We are now going to derive explicit formulae for (6.2). We will do this assuming (by induction) that the two As-operations commute on subgraphs of G. After that it will not be difficult to check the commutativity on G itself. Note that the recursion is correct because whatever property one wishes to prove for a graph, one only has to make assumptions about its subgraphs.

§ 7.1. Some notations.

Operation $\mathcal{R}$ on integrands. So far we have been using the operation $\mathcal{R}$ defined on integrated diagrams. But since now we will have to work with integrands, it is convenient to use the same symbol $\mathcal{R}$ to denote the operation of UV subtractions *prior* to integrations over p:

$$\mathcal{R} \overset{\text{def}}{=} 1 - \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa. \tag{7.1}$$

This operation is defined on graphs G as well as on the products of the form $\mathcal{P}_a(p_\gamma) \times \gamma(p_\gamma)$ where $\mathcal{P}_a(p_\gamma)$ is a polynomial while γ is any (κ- or m-) subgraph of G.

$\mathcal{R}$ associated with $\tilde{\mathbf{R}}$. Although the entire arbitrariness in the definition of $\mathcal{R}$ for an individual diagram G is in the operator $\tilde{\mathbf{r}}$ (more precisely, $\tilde{\mathbf{r}}_{(G)}$), it is more convenient to think about $\tilde{\mathbf{r}}$ in terms of the operation $\tilde{\mathbf{R}}$ (which also involves operators $\tilde{\mathbf{r}}_{(\gamma)}$ for $\gamma < G$). This is because all the explicit expressions for $\mathcal{R}$ (cf. eq.(3.1)) involve $\tilde{\mathbf{R}}$. On the other hand, the two points of view are equivalent if one recalls that in the problems of perturbative quantum field theory one deals with the entire universum of graphs: specifying the family of operations $\tilde{\mathbf{R}}_{(G)}$ on the entire universum of graphs G is equivalent to specifying the family of operators $\tilde{\mathbf{r}}_{(G)}$. Then, to fix an operation $\mathcal{R}$ (or, equivalently, $\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa$) on a hierarchy of graphs, it is sufficient to fix an operation $\tilde{\mathbf{R}}$ on it.

For the above reasons we will say that an operation $\mathcal{R}$ (and $\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa$) is *associated with* some operation $\tilde{\mathbf{R}}$, whenever it is necessary to indicate this kind of relationship between the two operations.

κ- and m-subgraphs. Since there are two expansions—in κ and m—in our problem that one has to deal with simultaneously, there are two systems of singular planes, complete subgraphs etc. in the same graph G.[14] In order to distinguish the objects from the two systems, we will call them κ- and m-subgraphs etc.

We will retain the symbols $\tilde{\mathbf{R}}$ and $\mathcal{R}$ for the operations associated with κ-singularities, and use the symbols $\tilde{\mathbf{R}}^m$ and $\mathcal{R}^m$ for the operations associated with m-singularities.

By default, a subgraph in a formula is κ-subgraph. Presence of m-subgraphs will be explicitly indicated.

The relation between the two systems of subgraphs is based on a simple principle: any factor $g \in G$ which is "m-singular" (i.e. develops singularities after expansion in m) is automatically "κ-singular"—because the set of momenta and masses with respect to which the expansion is done in the latter case comprises all such parameters in the former case. Thus, to every m-subgraph Γ there corresponds a unique κ-subgraph H obtained from Γ by "κ-completion".

§ 7.2. General formula for the expansion $\mathbf{As}_m\circ\mathcal{R}\circ G$.

Let us first explain the structure of (6.8). The dependence on p, m and M in (6.8) can be described as follows:

$$\mathbf{As}'_\kappa\circ G(p, M, m) = \sum D(p) \times C(M, m), \tag{7.2}$$

which correponds to the expansion (6.3). It is clear that $D(p)$ are distributions well-defined everywhere except for the point $p = 0$. The action of $\mathbf{As}_m$ in (6.5) can be described as

$$\mathbf{As}_m\circ\mathbf{As}'_\kappa\circ G(p, M, m) \equiv \sum D(p)\ \mathbf{As}_m\circ C(M, m) = \sum D(p)\ A(M)\ B(m) \tag{7.3}$$

(the explicit fromulae for $\mathbf{As}_m\circ C$ are yet to be determined). On the other hand,

$$\mathbf{As}_m\circ G(p, M, m) = \sum C'(p, M)\ B'(m) \tag{7.4}$$

[14] In general, one should also consider the third system of subgraphs—the one corresponding to singularities of unrenormalized non-expanded integrand $G(p, \ldots)$ that were discussed in §6.1. In order to keep notations simple, we agreed not to indicate explicitly the possible presence of such singularities, the corresponding operation $\tilde{\mathbf{R}}$ etc.

(cf. the explicit expression below in (7.7)), and

$$\mathbf{As}'_\kappa \circ \mathbf{As}_m \circ G(p,M,m) \equiv \sum \mathbf{As}'_\kappa \circ C'(p,M)\; B'(m) = \sum D'(p)\; A'(M)\; B'(m). \tag{7.5}$$

So far we don't know the form of C' and how $\mathbf{As}'_\kappa$ does its job on C' (which will be explained below). Nevertheless, the commutativity (6.8) implies that $D = D'$ etc.[15] It follows immediately that the commutativity is preserved if one replaces $\mathbf{As}'_\kappa$ by $\tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa$:

$$\mathbf{As}_m \circ \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa = \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa \circ \mathbf{As}_m. \tag{7.6}$$

Indeed, on both sides of (7.6) the operator $\tilde{\mathbf{r}}$ acts—as demonstrated above—on exactly the same p-dependent distributions $D(p)$.

The explicit version of (7.4) is analogous to (6.3) (cf. eqs.(3.1)–(3.3)):

$$\mathbf{As}_m \circ G(p,M,m) = \sum_{\substack{\varnothing \le \Gamma < G \\ \Gamma \text{ is } m\text{-subgraph}}} \sum_a \tilde{\mathbf{R}}^m \circ \big[\delta_a(p_\Gamma)\, \mathbf{T}_m \circ G \backslash \Gamma(p,M,m)\big] \times \mathcal{R}^m \circ \langle \mathcal{P}_{a,\Gamma} * \Gamma \rangle, \tag{7.7}$$

where

$$\mathcal{R}^m \circ \langle \mathcal{P}_{a,\Gamma} * \Gamma \rangle \equiv \langle \mathcal{P}_{a,\Gamma} * \mathcal{R}^m \circ \Gamma \rangle \equiv \lim_{\Lambda \to 0} \int dp'_\Gamma\, \Phi^\Lambda(p)\, \mathcal{P}_{a,\Gamma}(p'_\Gamma)\, \mathcal{R}^m \circ \Gamma(p'_\Gamma, m). \tag{7.8}$$

Note that since Γ is an m-subgraph of G, the object (7.8) is an almost-uniform (in the sense of [1], subsec.1.2) function of m and is independent of M and p. On the other hand, all the dependence on p is concentrated in the square-bracketed factor on the r.h.s. whose m-dependence is trivial while the M-dependence is not.

We now wish to apply the operation $\mathcal{R} = 1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa$ termwise to the above expansion (similarly to (7.5)). Owing to (7.6) this is equivalent to $\mathbf{As}_m \circ \mathcal{R} \circ G$. Since As-expansion of a product is a product of As-expansions [1] and since the only non-trivial dependence on κ is via M in $G \backslash \Gamma(p,M,m)$, one has:

$$\begin{aligned} &\mathbf{As}_m \circ \mathcal{R} \circ G(p,M,m) \\ &= \sum_{\substack{\varnothing \le \Gamma < G \\ \Gamma \text{ is } m\text{-subgraph}}} \sum_a \mathcal{R} \circ \tilde{\mathbf{R}}^m \circ \big[\delta_{a,\Gamma}(p_\Gamma)\, \mathbf{T}_m \circ G \backslash \Gamma(p,M,m)\big] \times \mathcal{R}^m \circ \langle \mathcal{P}_{a,\Gamma} * \Gamma \rangle. \end{aligned} \tag{7.9}$$

The action of $\mathcal{R}$ on the somewhat unusual expression in square brackets will be explained in the next subsection.

The above eq.(7.9) constitutes the final result of the analysis of the problem of Euclidean asymptotic expansions of Feynman diagrams as seen from the point of view of the abstract theory of As-expansions of products of singular functions. Below in section 8 we will transform it—using specific properties of Feynman diagrams proper—to a more convenient form similar to the As-operation as presented in [11]. The immediate remarks to be made here are as follows.

(i) The non-analytic dependences on the heavy parameters M and the light parameters m are clearly factorized in (7.9). Indeed, the expression in angle brackets is independent of M by construction. On the other hand, the M-dependent distributions over p in square brackets are pure power series in m (due to the action of $\mathbf{T}_m$ and the fact that neither $\tilde{\mathbf{R}}^m$ nor $\mathcal{R}$ affect the resulting powers of m). Therefore, it is clear—in the context of ordinary short-distance OPE—that the angle-bracketed expressions correspond to matrix elements of OPE (the polynomials $\mathcal{P}$ then correspond to vertices with composite operators) while the square-bracketed expressions, to coefficient functions.

(ii) All the quantities in (7.9) are finite by construction: the angular-bracketed "matrix element" has its UV divergences removed by $\mathcal{R}^m$, while the IR and UV singularities of the square-bracketed "coefficient functions" are eliminated by $\tilde{\mathbf{R}}^m$ and $\mathcal{R}$, respectively.

[15] Strictly speaking D etc. depend on a summation index so that one may need to take linear combinations to establish the equality.

(*iii*) The expression (7.9) as a whole is independent of the specific choice of the operation $\tilde{\mathbf{R}}^m$ and the associated operation $\mathcal{R}^m$. This is a usual feature of representation of an As-expansion in terms of an intermediate $\tilde{R}$-operation (recall that an As-expansion is unique [10], [1]).

§ 7.3. Defining $\tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa \circ \tilde{\mathbf{R}}^m \circ [\ldots]$ in (7.9).

Consider the following object from (7.9) (p'_Γ is the integration momentum implied in $\langle \ldots \rangle$; it is isomorphic to p_Γ):

$$\sum_a \delta_a(p_\Gamma)\, \mathbf{T}_m \circ G \backslash \Gamma(p, M, m) \times \mathcal{P}_{a,\Gamma}(p'_\Gamma) \qquad (\Gamma \text{ is } m\text{-subgraph}). \tag{7.10}$$

When δ_a is integrated out, its derivatives also affect $G\backslash\Gamma$. Let us exhibit this explicitly using the fact that the above pattern of the polynomials $\mathcal{P}_a$ and the δ-functions δ_a is characteristic of the operation of Taylor expansion (cf. eq.(9.6) in [1]). We use the following elementary identity:

$$\sum_a \delta_a(p_\Gamma) F(p_\Gamma) \times \mathcal{P}_a(p'_\Gamma) = \sum_b \delta_b(p_\Gamma) \times \mathbf{T}_{p'_\Gamma} \circ F(p'_\Gamma)\, \mathcal{P}_b(p'_\Gamma). \tag{7.11}$$

Now split the variable p as

$$p = (p_\Gamma, p_{G\backslash\Gamma}),$$

and define

$$m_\Gamma = (m, p_\Gamma), \qquad\qquad m'_\Gamma = (m, p'_\Gamma). \tag{7.12}$$

Then we can rewrite (7.10) as

$$\text{eq.(7.10)} = \sum_b \delta_b(p_\Gamma) \times \left[\mathbf{T}_{m'} \circ G\backslash\Gamma(p_{G\backslash\Gamma}, M, m')\right] \mathcal{P}_b(p'_\Gamma). \tag{7.13}$$

The variables $p_{G\backslash\Gamma}$ on the r.h.s. parameterize the singular plane π_Γ on which the entire expression is localized. Note that $G\backslash\Gamma$ is being expanded in p'_Γ, so that all the singularities are with respect to $p_{G\backslash\Gamma}$.

The above formula allows one to define $\tilde{\mathbf{R}}^m$ on such expressions (cf. the reasoning in sect.3 above and §14.2 of [1])) as follows:

$$\sum_a \tilde{\mathbf{R}}^m \circ \left[\delta_a(p_\Gamma)\, \mathbf{T}_m \circ G\backslash\Gamma(p, M, m)\right] \times \mathcal{P}_{a,\Gamma}(p'_\Gamma) \qquad (\Gamma \text{ is } m\text{-subgraph}) \tag{7.14}$$

$$\stackrel{\text{def}}{=} \sum_b \delta_b(p_\Gamma) \times \tilde{\mathbf{R}}^m \circ \mathbf{T}_{m'} \circ G\backslash\Gamma(p_{G\backslash\Gamma}, M, m')\, \mathcal{P}_b(p'_\Gamma)$$

$$= \delta(p_\Gamma) \times \tilde{\mathbf{R}}^m \circ \mathbf{T}_{m'} \circ G\backslash\Gamma(p_{G\backslash\Gamma}, M, m') + \ldots,$$

where on the r.h.s. we have not shown the terms proportional to derivatives of the δ-function—such derivatives vanish after integration over p which eventually has to be done. Note that there need not be any correlation between the definitions of the operation $\tilde{\mathbf{R}}^m$ on $\mathbf{T}_m G$ and on $\mathbf{T}_{m'} \circ G\backslash\Gamma$ (cf. the reasoning in section 3).

In (7.9), the expression (7.14) is being acted on by $\mathcal{R} = 1 - \tilde{\mathbf{r}} \circ \mathbf{As}'_\kappa$. Here one only has to deal with the non-trivial dependence on M on the r.h.s., and it is easy to understand that $\mathbf{As}'_\kappa$ should be applied termwise to the r.h.s., effectively getting combined with $\tilde{\mathbf{R}}^m$.[16] It remains to note that, as usual, the operator $\tilde{\mathbf{r}}$ on

[16] In [1] the formulae for the As-operation were presented only for a class of singular functions without non-integrable singularities prior to expansion. In the present case, we for a first time encounter a situation where an As-operation—$\mathbf{As}'_\kappa$—is being applied to an expression (7.14) which involves an $\tilde{R}$-operation. As was noted in [1], extension of the formula for As-operation to the case of a singular initial expression is straightforward (see also below §7.4).

$\mathbf{As}'_\kappa\circ\tilde{\mathbf{R}}^m\circ\mathbf{T}_{m'}\circ G\backslash\Gamma$ (or, equivalently, $\tilde{\mathbf{R}}$ on the $\mathbf{T}_\kappa\circ\mathbf{T}_{m'}\circ G\backslash\Gamma \equiv \mathbf{T}_{\kappa'}\circ G\backslash\Gamma$, where $\kappa' = (\kappa, p'_\Gamma)$) can be chosen independently from $\tilde{\mathbf{r}}$ on $\mathbf{As}'_\kappa\circ\tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ G$ (cf. section 3).

Finally, eq.(7.9) takes the form:

$$\mathbf{As}_m\circ\mathcal{R}\circ G(p,M,m) \tag{7.15}$$
$$= \sum_{\substack{\emptyset\le\Gamma<G \\ \Gamma \text{ is } m\text{-subgraph}}} \delta(p_\Gamma)\times\mathcal{R}\circ\int dp'_\Gamma\,\tilde{\mathbf{R}}^m\circ\mathbf{T}_{m'}\circ G\backslash\Gamma(p_{G\backslash\Gamma},M,m')\,\mathcal{R}^m\circ\Gamma(p'_\Gamma,M,m)+\ldots.$$

Note that $G\backslash\Gamma$ depends on p'_Γ through m'. It should also be remembered that the integration over p'_Γ should be understood in the sense of the principal value (operation $*$).

Eq.(7.15) eliminates the last unknown in (7.9) and represents a convenient starting point for studying exponentiation of As-operation on collections of Feynman diagrams corresponding to Green functions (section 8 below).

§ 7.4. As-operation on products involving $\tilde{\mathrm{R}}$ and δ-functions. Proof of commutativity of the two As-operations.

Both in (7.9) and in (7.15) one has to consider As-expansions of products involving an $\tilde{R}$-operation and/or δ-functions. Therefore, it is worthwhile to consider this point from a general point of view. Moreover, it turns out that the object

$$\tilde{\mathbf{R}}^m\circ\big[\delta_{a,\Gamma}(p_\Gamma)\,\mathbf{T}_m\circ G\backslash\Gamma(p,M,m)\big] \tag{7.16}$$

from (7.9) can be analyzed without performing projections onto the plane singled out by the $\delta_{a,\Gamma}(p_\Gamma)$—in complete analogy with expressions of the form $\tilde{\mathbf{R}}^m\circ G$. The simplicity and straightforward character of the resulting formal proof of commutativity (7.6) is another example of how the meticulous attention to the formalism and notations in the preceding paper [1] pays off.

Indeed, with some experience with the formalism and understanding of the mechanism of the As-operation, an explicit expression for $\mathcal{R}$ on the distribution in square brackets in (7.9) can be written down offhand. In view of (7.1), it is sufficient to present expressions for the operation $\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa$.

To start, recall the expression for $\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G$ (6.3), which we here rewrite in slightly different notations similar to those used above:

$$\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,M,m) = \sum_{\substack{\emptyset\le\gamma<G \\ \gamma \text{ is } \kappa\text{-subgraph}}}\ \sum_b \tilde{\mathbf{R}}\circ\big[\delta_b(p_\gamma)\,\mathbf{T}_\kappa\circ G(p,M,m)\big]\times\mathcal{R}\circ\langle\mathcal{P}_b*\gamma\rangle. \tag{7.17}$$

First we wish to write down a similar expression for the product G replaced as

$$G(p,M,m)\to G'(p,M,m) = \tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ G(p,M,m). \tag{7.18}$$

To put it simply, some of the factors have been replaced (owing to the action of $\mathbf{T}_m$) by arbitrarily singular factors; on top of everything, the singularities of the resulting expression have been subtracted using $\tilde{\mathbf{R}}^m$. Expressions of this form appear in (7.15).

Second, we also wish to consider expressions obtained from (7.18) by replacing the group of factors corresponding to the m-subgraph Γ by one factor—the δ-function δ_a:

$$G'(p,M,m)\to G''(p,M,m) = \tilde{\mathbf{R}}^m\circ\big[\delta_a(p_\Gamma)\,\mathbf{T}_m\circ G\backslash\Gamma(p,M,m)\big]. \tag{7.19}$$

The crucial thing to realize is that neither the philosophy nor the reasoning of the theory of As-operation developed in [1] need to be changed in order to deal with such products. Indeed, as to the singularities and the additional R-operation (the operation $\tilde{\mathbf{R}}^m$ in our case), one only has to bear in mind the following. As a "formal

expansion" one should take the original product *without* the additional R-operation but with the factors that can be expanded, expanded. One then proceeds to constructing the counterterms (introducing an intermediate $\tilde{R}$-operation etc.) treating on an equal footing both the "old" singularities (i.e. corresponding to the factors that are singular prior to expansion) and those generated by formal expansion. However, in constructing the counterterms via consistency conditions, one uses the initial expression which contains *both* non-expanded factors *and* the additional R-operation. The entire procedure becomes perfectly obvious if one recalls the philosophy of constructing the expansion by considering it first in an open region in the space of p where all the factors are regular and then expanding the domain of definition of the expansion by adding counterterms proportional to δ-functions. By analogy with (7.17), one immediately obtains a similar expression for G':

$$\begin{aligned}&\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ\tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ G(p,M,m)\\ &\quad=\sum_{\substack{\varnothing\leq\gamma<G\\ \gamma\text{ are }\kappa\text{-subgraphs}}}\sum_b\tilde{\mathbf{R}}\circ\big[\delta_b(p_\gamma)\mathbf{T}_\kappa\circ G\backslash\gamma(p,M,m)\big]\times\mathcal{R}\circ\langle\mathcal{P}_b*\tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ\gamma(p,M,m)\rangle.\end{aligned}\tag{7.20}$$

Note that we have replaced $\mathbf{T}_\kappa\circ\mathbf{T}_m$ acting on $G\backslash\gamma$ by $\mathbf{T}_\kappa$.

Turning to the expression (7.19), it is not difficult to realize that the δ-function is as good as any other factor from the point of view of the formalism—provided one considers it as a singular factor. Its only effect is that now every κ-subgraph h must contain the δ-function plus, perhaps, some other κ-singular factors. A simple combinatorial observation is that the subgraphs h are in one-to-one correspondence with κ-subgraphs γ of G such that $\gamma\geq\Gamma$. Without more ado we get:

$$\begin{aligned}&\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ\tilde{\mathbf{R}}^m\circ\big[\delta_a(p_\Gamma)\,\mathbf{T}_m\circ G\backslash\Gamma(p,M,m)\big]\\ &\quad=\sum_{\Gamma\leq\gamma<G}\sum_b\tilde{\mathbf{R}}\circ\big\{\delta_b(p_\gamma)\,\mathbf{T}_\kappa\circ G\backslash\gamma(p,M,m)\big\}\\ &\qquad\times\mathcal{R}\circ\langle\mathcal{P}_b*\tilde{\mathbf{R}}^m\circ\big[\delta_a(p_\Gamma)\,\mathbf{T}_m\circ\gamma\backslash\Gamma(p,M,m)\big]\rangle.\end{aligned}\tag{7.21}$$

One should remember that in this expression γ are κ-subgraphs in G, while Γ are m-subgraphs. Summation runs over γ in accordance with the above observation about one-to-one correspondence between κ-subgraphs in G and in the product G'' given by (7.19).

It is a good excercise to verify equivalence of (7.21) and the expressions that can be obtained along the lines of the preceding subsection.

It remains to note that now one can easily check the formal commutativity of the two As-operations (7.6) by substituting (7.20) into the r.h.s. of (7.6) and using the expression for its l.h.s. that has already been discussed—provided $\mathbf{As}_m$ on the l.h.s. can be put under integration. The final justification for the latter comes from inequalities of the type (6.12) that will be obtained in section 9.

8. Diagrammatic form of $\mathbf{As}_m\circ\mathcal{R}\circ G$.

The formulae derived in the preceding section are not immediately useful for deriving As-expansions for Green functions in OPE-like form. To this end one should recast eq.(7.15) into a form that would take into account specific properties of Feynman diagrams that were irrelevant at the analytic stage.

The diagrammatic analysis of Euclidean As-expansions was performed in much detail in [11]. The combinatorial aspects which we are going to discuss in this section are not very sensitive to whether one deals with expansions in dimensionally regularized form, as in [11], or in formalism without regularizations, as in the present paper. Therefore, we will give only an outline of the reasoning and consider just two key examples: the ordinary short-distance OPE and the expansion in heavy masses (which extends the decoupling theorem of Appelquist and Carrazzone—for a review see [20]). An interested reader can find further examples and a detailed description of the combinatorial techniques in [11].

§ 8.1. Using factorization properties of $G\backslash\Gamma$.

At this point we may suppose that G is an ordinary Feynman diagram and use the factorization properties of the expression $G\backslash\Gamma(p, M, m)$.[17]

An m-subgraph Γ is any set of lines and vertices of G such that every line from Γ is singular after expansion in m (irrespective of whether or not the line is singular before expansion); Γ must also satisfy the completeness condition (see §7.2 in [1]). In the present case completeness of Γ means the following:

(i) When one nullifies all light external momenta from the set m as well as all the momenta flowing through the lines of Γ, no other m-singular line of G will have its momentum nullified owing to momentum conservation at vertices;

(ii) Γ contains all those and only those vertices of G whose all incident lines belong to Γ; no heavy external momentum from the set M is allowed to enter into such a vertex.

Consider the complement of Γ in G, denoted as $G\backslash\Gamma$. The graphical image for $G\backslash\Gamma$ is obtained by deleting the lines and vertices belonging to Γ from the diagram G. We have already encountered a similar situation in sect.3, where $G\backslash\Gamma$ decomposed into a set of 1PI UV-subgraphs. Consider the connected components of $G\backslash\Gamma$, denoting them generically as ξ:

$$G\backslash\Gamma = \prod_i \xi_i.$$

Denote the set of the loop momenta that are internal with respect to ξ as p_ξ (ξ may have no loops at all in which case p_ξ is empty; this has no effect on our formalism). Then the variable p can be decomposed as follows:

$$p = (p_\Gamma, p_1, \ldots, p_i, \ldots).$$

Then

$$G\backslash\Gamma(p_{G\backslash\Gamma}, M, m, p_\Gamma) = \prod_i \xi_i(p_i, M, m_i),$$

where we have introduced the notation m_i for the collection of parameters which contains those light parameters from m, as well as those components of p_Γ, on which ξ_i depends.

The As-expansion we are dealing with is independent of the choice of the operation $\tilde{\mathbf{R}}^m$. in particular, we may fix $\tilde{\mathbf{R}}^m$ to be factorized in the sense of section 11 of [1]. Then the associated operation $\mathcal{R}$ will be also factorized (see Appendix A). Therefore, performing integration over p, we obtain instead of (7.9) the following expression:

$$\mathbf{As}_m\circ\mathcal{R}\circ\int dp\, G(p, M, m) = \sum_{\substack{\varnothing\leq\Gamma<G \\ \Gamma \text{ is } m\text{-subgraph}}} \mathcal{R}^m\circ\int dp_\Gamma \left(\prod_i \Delta^{\mathrm{as}}\circ\xi_i(M, m_i)\right)\Gamma(p_\Gamma, m), \tag{8.1}$$

where:

$$\Delta^{\mathrm{as}}\circ\xi_i(M, m_i) = \mathcal{R}\circ\int dp_i\, \tilde{\mathbf{R}}^m\circ\mathbf{T}_{m_i}\circ\xi_i(p_i, M, m_i). \tag{8.2}$$

(Note that only the "counterterms" Δ are sensitive to the operation $\mathcal{R}$ in the initial diagram, while the operation $\mathcal{R}^m$ used to perform UV subtractions on the r.h.s. of (8.1) is associated (in the sense of §7.1) with the operation $\tilde{\mathbf{R}}^m$ used to subtract IR singularities from the formal expansion on the r.h.s. of (8.2). Recall that p_i are loop momenta of ξ_i.)

A "fool-proof" recipe for enumeration of subgraphs in (8.1). It is interesting to note, following [11], that the condition of m-completeness of Γ admits a universal and very convenient "fool-proof" reformulation. The above formulae will remain correct if, instead of summing over m-subgraphs Γ, one performs summations

[17] The reasoning below follows section 3 which in turn is reminiscent of [11].

over all collections of pairwise non-intersecting and otherwise arbitrary subgraphs ξ. Then in order to nullify superfluos terms it is sufficient to demand that

(i) whenever the operation $\mathbf{T}_m$ generates meaningless expressions of the type $1/0$ (due to a propagator carrying only a combination of light external momenta) Δ_i should be put equal to zero;

(ii) if setting $M = \infty$ in Γ produces a factor $1/\infty$ in denominator (due to a heavy line that happened to remain outside all ξ_i's) then such terms should be put to zero in the sum in (8.1).

Such a reformulation is very convenient for studying exponentiation of expansions of Green functions.

Interpreted graphically, eq.(8.1) corresponds to shrinking the subgraphs ξ_i to vertices to which there correspond the factors (8.2) which are polynomials of the momenta entering the new vertices.

The two formulae (8.1) and (8.2) represent a fundamental explicit expression for the As-operation on a renormalized Feynman diagram.

§ 8.2. Exponentiation of the As-operation into an OPE-like form.

The two expressions (8.1) and (8.2) have exactly the same combinatorial structure as that of the As-expansions in the dimensionally regularized form studied in [11]. As was pointed out there, similarity of their structure to that of the R-operation in the MS scheme allows one to easily obtain expansions for entire collections of diagrams corresponding to Green functions in the global OPE-like form. In fact, the situation here is even simpler than in [11] because now all the terms in the expansion (8.1) which is a starting point for studying exponentiation, are finite. Therefore, one need not perform the step of inversion of the R-operation, which was the most cumbersome part of [11] (the role of inverted R-operation is played by the operation $\tilde{\mathbf{R}}^m$ in (8.2)). Repeating the reasoning of [11] *mutatis mutandis* one can immediately write down exponentiated forms for expansions of Green functions.

Recall that for each asymptotic regime one only has to find, starting from the basic definitions of the m-subgraph Γ, diagrammatic characterization of the connected components ξ_i of its complement $G\backslash\Gamma$—wherein the above "fool-proof" enumeration recipe is very convenient.

(i) Consider the case corresponding to the short-distance OPE. Then one has only two (one independent—after taking into account momentum conservation) heavy external momenta, while all the masses are considered as light parameters. One finds:

$$\mathcal{R}\circ\langle T\Big\{\int dx e^{iqx}\, A(x)B(0)\, \exp i[\mathcal{L}+\varphi J]\Big\}\rangle_0 \tag{8.3}$$

$$\underset{q^2\to-\infty}{\simeq} \sum_i C_i(q)\, \mathcal{R}^m\circ\langle T\big\{O_i(0)\, \exp i[\mathcal{L}+\varphi J]\big\}\rangle_0,$$

with the coefficient functions C_i specified by the following expression:

$$\sum_i C_i(q)\, O_i(0) = \mathcal{R}\circ\tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ\int dx e^{iqx}\, T\big\{A(x)B(0)\, \exp i\mathcal{L}\big\}^{\mathrm{conn}}. \tag{8.4}$$

To correctly interpret these expressions one should keep in mind that the standard perturbative formalism of interaction representation is used here. Thus, A, B and O_i are local monomials of free fields (*without* normal ordering) while radiative corrections are generated by the chronological (T-) exponents of the interaction Lagrangian $\mathcal{L}$ (integration over the space time is included into $\mathcal{L}$). UV renormalization is performed by the operations $\mathcal{R}^m$ and $\mathcal{R}$. The operation $\mathbf{T}_m$ acts as follows: one expands the T-product on the r.h.s. of (8.4) in Wick normal products of the free fields, retains only connected diagrams that cannot be divided into disconnected parts by cutting any one of propagators corresponding to the light fields (cf. the above "fool-proof" recipe) and the operation $\mathbf{T}_m$ expands the resulting loop diagrams both in masses and the momenta corresponding to the free fields in normal products.

Individual coefficient functions C_i can be extracted by taking corresponding matrix elements of both sides of (8.4) (only tree-level diagrams will be present on the l.h.s. since there is no T-exponentiated Lagrangian there to generate radiative corrections). Such a procedure is analogous to the algorithm of calculating coefficient functions of OPE in the MS scheme described in [8].

We see that our formulae are in direct correspondence with the formulae and algorithms developed in [7], [8], [9], [10], [11]. This should be no wonder because the methods we use were developed from the very beginning as a straightforward formalization of the reasoning of those papers.

It is also worth stressing that our formalism contains nothing similar to the oversubtraction techniques of [25] (see also [13] for a discussion of this point).

(*ii*) As a second example, consider the case when the set of heavy parameters M consists of only heavy masses. Consider the generating functional of Green functions of light particles:

$$\mathcal{R}\circ\langle T\ \exp[i\mathcal{L}(\varphi,\Phi)+\varphi J]\rangle_0, \tag{8.5}$$

where $\mathcal{L}(\varphi,\Phi)$ is the total (integrated over space-time) Lagrangian of the system which depends on heavy and light particles. Supposing that typical momenta of φ and the masses of the light fields m are of the same order of magnitude and much less than the masses M of the heavy fields, one obtains:

$$\text{eq.}(8.5)=\mathcal{R}^m\circ\langle T\ \exp[i\mathcal{L}_{\text{eff}}(\varphi)+\varphi J]\rangle_0, \tag{8.6}$$

where the effective low-energy Lagrangian whose expression is similar to (8.4):

$$\begin{aligned} i\mathcal{L}_{\text{eff}}(\varphi) &= \mathcal{R}\circ\tilde{\mathbf{R}}^m\circ\mathbf{T}_m\circ\big\{T\exp i\mathcal{L}-1\big\}^{\text{light}-1\text{PI}} \qquad (8.7)\\ &\equiv \sum_n g_{n,\text{eff}}(M)\int dxO_n(x), \end{aligned}$$

where $g_{n,\text{eff}}(M)$ are the couplings of the effective Lagrangian. Note (cf. [11]) that $\mathcal{L}_{\text{eff}}$ can contain contributions that are quadratic in the light fields. This corresponds to the finite M-dependent field renormalization in the usual formulation of the decoupling theorem (for a review see [20]. Also note that only analytic dependence on the light masses is allowed in $\mathcal{L}_{\text{eff}}$. "light-1PI" means (cf. the above "fool-proof" recipe) that one has to take into account only diagrams that have no heavy external fields and such that they cannot be divided into two disconnected pieces by cutting a line corresponding to a light particle.

This completes our discussion of the structure of Euclidean asymptotic expansions of Feynman diagram within the formalism of the As-operation.

9. Double As-expansions: existence and properties.

The aim of the present section is to prove a theorem on double As-expansions which summarizes all analytic facts that are necessary for derivation of Euclidean asymptotic expansions of renormalized Feynman diagrams. Explicit formulae have already been presented in section 7.

We will perform the reasoning in an abstract manner of [1], without explicit mentioning of Feynman diagrams proper. All the analytic formulae here—however cumbersome in appearance—are based on a primordially primitive power counting. The apparent abstruseness is due to presence of two expansion parameters and another one used to describe singularities—each of the three accompanied by an integer-valued index etc. Nevertheless, the powerful formalism of [1] allows one to use the recursive structures inherent in the problem and cut through all the complexities of the proof in an explicit algebraic fashion.

An abstract mathematical character of the following text makes it necessary to recycle some of the physics-inspired notations used in the preceding sections: the symbols m and M—alongside of n and N—will be used for interger-valued indices while the two expansion parameters will be denoted as κ and σ. We will not need M in its old meaning. Other notations follow [1].

§ 9.1. Double As-expansions.

We start with a formal definition of double As-expansion and present a simple lemma, which can be considered as a generalization of the uniqueness property to the case of double As-expansions.

Let $G(\kappa,\sigma)$ be a distribution which depends parametrically on two real parameters κ and σ from a rectangle $(0,\kappa_0)\times(0,\sigma_0)$.

Suppose there exist asymptotic expansions of G in powers and logs of the parameters κ and σ. In the notations of [1] the sum of terms of order κ^n is denoted as $\mathbf{as}^n_\kappa\circ G$ and the partial sum of the terms through the power n, as $\mathbf{As}^n_\kappa\circ G$ (and similarly for σ). By the definition of As-expansion the following asymptotic estimate must be fulfilled for all σ:

$$(1-\mathbf{As}^n_\kappa)\circ G = o(\kappa^n).$$

Each term of the expansion is a distribution parametrically depending on σ. Assume that there exist As-expansions of those distributions in σ. Denote the double series thus obtained as $\mathbf{As}_\sigma\circ\mathbf{As}_\kappa\circ G$. We can reverse the order of expansions and ask a natural question, whether the two resulting double series $\mathbf{As}_\sigma\circ\mathbf{As}_\kappa\circ G$ and $\mathbf{As}_\kappa\circ\mathbf{As}_\sigma\circ G$ coincide. Generally speaking, they don't (the simplest example is the numeric function $1/(\kappa+\sigma)$). However, it is possible to formulate a necessary condition for the commutativity of the two As-operations based on the notion of *double As-expansion*.

Consider the *double remainder* $\langle(1-\mathbf{As}^n_\kappa)\circ(1-\mathbf{As}^m_\sigma)\circ G,\varphi\rangle$. By definition, it is $o(\kappa^n)$ for all σ, but its behaviour as $\sigma\to 0$ is a priori unpredictable. It is natural to introduce the following definition:

Definition. A double series in powers and logs of κ and σ—its partial sum of terms through $O(\kappa^n)\times O(\sigma^m)$ is denoted as $\mathbf{As}^{n,m}_{\kappa,\sigma}\circ G$—is called *double As-expansion* if:

$$\left|\langle(1-\mathbf{As}^n_\kappa-\mathbf{As}^m_\sigma+\mathbf{As}^{n,m}_{\kappa,\sigma})\circ G,\varphi\rangle\right| < o(\kappa^n)\,o(\sigma^m),$$

and there exist integers n_0, m_0 such that $\mathbf{as}^{n,m}_{\kappa,\sigma}\circ G=0$, provided $n\le n_0$ or $m\le m_0$.

One can see that the double As-expansion is unique. Moreover, its existence implies nice properties of the double series obtained by termwise composition of the two one-parameter As-expansions like $\mathbf{As}_\kappa\circ\mathbf{As}_\sigma\circ G$, which can be summarized in the following elementary lemma:

Lemma 1. If there exists a double As-expansion of the graph G then there exist series $\mathbf{As}_\kappa\circ\mathbf{As}_\sigma\circ G$ and $\mathbf{As}_\sigma\circ\mathbf{As}_\kappa\circ G$; moreover:

$$\mathbf{As}_\sigma\circ\mathbf{As}_\kappa\circ G=\mathbf{As}_\kappa\circ\mathbf{As}_\sigma\circ G=\mathbf{As}_{\kappa,\sigma}\circ G.$$

Therefore, to prove the commutativity of the two As-operations it is sufficient to construct the double As-expansion—which is the purpose of the rest of this section.

§ 9.2. Object of expansion.

The objects we are working with are the graphs, formally defined in sections 4 and 18 of [1]. A graph in this sense is an abstraction to describe products of singular functions encountered on a regular basis in studies of multiloop diagrams (e.g. integrands of multiloop diagrams in momentum-representation). As we now wish to study expansions in two parameters, the notations of [1] should be extended. Namely, the linear functions $l_g(p)$ (which describe the way the integration (loop) momenta p are combined in the argument of the g-th factor) is now required to have the form:

$$l_g(p)=l'_g(p)+\kappa\sigma l''_g+\kappa l'''_g, \tag{9.1}$$

where l''_g and l'''_g represent linear combinations of small and large external momenta, respectively, and are independent of p. Some of the functions F_g which used to depend on the expansion parameter κ, now acquire

dependence on the second expansion parameter σ of the form:

$$F_g(q,\kappa) \to F_g(q,\kappa\sigma), \tag{9.2}$$

i.e. instead of κ we now have the product $\kappa\sigma$. Otherwise, the properties of the functions F remain the same.

The assumptions (9.1)–(9.2) are crucial for existence of double As-expansion.

To simplify formulae, we assume that the formal expansions of F's start from $\kappa^0\sigma^0 = 1$, which can always be achieved by multiplying F by the corresponding powers of κ and σ.

We are going to show that the graph $G(p,\kappa,\sigma)$ has a double asymptotic expansion in powers and logarithms of κ and σ with the remainder bounded by an expression in which the dependences on κ and σ are factorized, i.e. that there exist double As-expansion of G.

The theorem on double asymptotic expansions presented below is, essentially, a logical outcome of the conditions (9.1)–(9.2). Heuristically, it is clear why this is so: a numeric function of the form $1/(1+\kappa+\kappa\sigma)$ with a structure analogous to (9.1), can be expanded into a double As-expansion in κ and σ. The latter property is naturally inherited by any products of such functions. The pathological cases of the sort mentioned in §9.1 are prohibited by the imposed restrictions.

Theorem 1. Under the above conditions, there exists a double asymptotic expansion of the graph $G(p,\kappa,\sigma)$; it is given by a termwise composition of the two As-operations as described in the preceding sections (see also below eqs.(9.5)–(9.6); and it has the following properties:

(a) $\mathbf{As}_\kappa \circ \mathbf{As}_\sigma \circ G \;=\; \mathbf{As}_\sigma \circ \mathbf{As}_\kappa \circ G \;=\; \mathbf{As}_{\kappa,\sigma} \circ G.$

(b) $\mathbf{As}^{n,m}_{\kappa,\sigma} \circ G = 0$ for $n < A^{\kappa G}$ or $m < A^{\sigma G}$, where

$$A^{\kappa G} = \max_{\substack{\Gamma \le G \\ \Gamma \text{ is } \kappa\text{-subgraph}}} (0, \omega_\Gamma), \qquad A^{\sigma G} = \max_{\substack{\Gamma \le G \\ \Gamma \text{ is } \sigma\text{-subgraph}}} (0, \omega_\Gamma).$$

(c) The operation $\mathbf{As}_{\kappa,\sigma}$ is local in the sense of [1].

(d) For the terms of the expansion the following estimate holds: for all $\varphi \in \mathcal{D}(P)$ such that $\operatorname{rad\,supp} \varphi \le d$,

$$\left|\langle \mathbf{as}^{n,m}_{\kappa,\sigma} \circ G, \varphi \rangle\right| \le \kappa^n \sigma^m d^{-\omega_G - n} \mathcal{S}[\varphi, d] \Lambda(\kappa,\sigma).$$

(Here and below we do not indicate the upper limits of summations since their exact expressions are cumbersome and of no practical use. They can, however, be restored from the proofs in a straightforward manner.)

(e) The remainder of the double expansion, defined as

$$\Delta_{n,m} \stackrel{\text{def}}{=} 1 - \mathbf{As}^n_\kappa - \mathbf{As}^m_\sigma + \mathbf{As}^{n,m}_{\kappa,\sigma},$$

satisfies the following "factorizable" estimate: one can fix a constant $C > 0$ such that for all $d > C\kappa$ and $\varphi \in \; D(P)$ with $\operatorname{rad\,supp} \varphi \le d$:

$$|\langle \Delta_{n,m} \circ G, \varphi \rangle| \le \kappa^{n+1} \sigma^{m+1} d^{-\omega_G - n - 1} \mathcal{S}[\varphi, d] \Lambda(\kappa,\sigma). \tag{9.3}$$

(f) The expansion possesses the following *minimality property*:

$$\langle \mathbf{as}^{n,m}_{\kappa,\sigma} \circ G * \mathcal{P}_{a,G} \rangle = \langle \Delta_{n,m} \circ G * \mathcal{P}_{a,G} \rangle = 0, \quad \text{for} \quad |a| \le \omega_{\scriptscriptstyle G} + n.$$

§ 9.3. Proof of theorem 1.

The proof of the theorem will be carried out by induction over the hierarchy of m-subgraphs $\Gamma < G$. It is convenient to include into the induction the following lemma containing useful auxiliary estimates:

Lemma 2.

(*i*) $\forall\varphi \in \mathcal{D}(\mathcal{P})$, $\operatorname{rad}\operatorname{supp}\varphi \leq d$:

$$|\langle \mathbf{as}^n_\kappa \circ (1 - \mathbf{As}^m_\sigma)\circ G\rangle| \leq \kappa^n \sigma^{m+1} d^{-\omega_G - n} \mathcal{S}[\varphi, d]\Lambda(\kappa, \sigma),$$

(*ii*) $\exists C > 0\ \forall d > C\kappa\ \forall\varphi \in \mathcal{D}(P)$, $\operatorname{rad}\operatorname{supp}\varphi \leq d$:

$$|\langle \mathbf{as}^m_\sigma \circ (1 - \mathbf{As}^n_\kappa)\circ G\rangle| \leq \kappa^{n+1} \sigma^m d^{-\omega_G - n-1} \mathcal{S}[\varphi, d]\Lambda(\kappa, \sigma).$$

The statements of the theorem and the lemma are trivial for the empty graph $G = 1$. Let us suppose that they hold for any subgraph of G. The proof can be divided into three logical steps. First, one defines As-expansion as a distribution on $\mathcal{D}(P\backslash\{0\})$ using decompositions of unit (cf. section 21 of [1]) and verify the conditions of the theorem for it. Second, one performs a natural extension of the distributions obtained at the first step onto the space of functions from $\mathcal{D}(P)$ with zero of the order $B_n = \omega_G + n + 1$ (such space is denoted as $\mathcal{D}_{B_n}(P)$). Third, one continues the As-expansion onto the entire $\mathcal{D}(P)$ and determines a finite renormalization to ensure asymptotic estimates.

It is convenient to carry out the first and second steps simultaneously.

Steps 1–2. To begin with, take a function $\varphi \in \mathcal{D}_{B_n}(P)$ and a cutoff $\phi_\lambda \in \mathcal{D}(P\backslash\{0\})$. Using the sector decomposition of unit we define $\mathbf{as}_{\kappa,\sigma}\circ G$ on $\varphi\phi_\lambda$:

$$\langle \mathbf{as}^{N,M}_{\kappa,\sigma}\circ G\,,\, \varphi\phi_\lambda\rangle = \sum_{\Gamma \triangleleft G} \sum_{\substack{n\leq N\\ m\leq M}} \langle \mathbf{as}^{n,m}_{\kappa,\sigma}\circ\Gamma\,,\, \mathbf{t}^{N-n}_\kappa \circ \mathbf{t}^{M-m}_\sigma \circ G\backslash\Gamma\,\theta_\Gamma\,\varphi\phi_\lambda\rangle.$$

Using the estimate (d) for $\Gamma < G$ (which holds by inductive assumptions) and the auxiliary estimate (B.1) we conclude that:

$$\left|\langle \mathbf{as}^{N,M}_{\kappa,\sigma}\circ G\,,\, \varphi\phi_\lambda\rangle\right| \leq \kappa^n\sigma^m \sum_{k\geq B_n} \lambda^{k-\omega_G - n}\|\varphi\|^k \Lambda(\lambda)\,\Lambda(\kappa,\sigma).$$

Integration over λ (cf. sect.21 of [1]) completes steps 1–2 for the estimate (d).

The estimate (i) is proved similarly using the auxiliary estimate (B.2).

The estimates (ii) and (e) are of a somewhat different kind, which should be clear from their look. First of all, we prove the following lemma:

Lemma 3. $\forall\varphi$, $\operatorname{rad}\operatorname{supp}\varphi \leq C\kappa$:

$$|\langle \mathbf{as}^m_\sigma \circ G\,,\, \varphi\rangle| \leq \sigma^m \kappa^{-\omega_G} \mathcal{S}[\varphi, \kappa]\Lambda(\sigma),$$

$$|\langle (1 - \mathbf{As}^m_\sigma)\circ G\,,\, \varphi\rangle| \leq \sigma^{m+1} \kappa^{-\omega_G} \mathcal{S}[\varphi, \kappa]\Lambda(\sigma).$$

The unusual way of how κ enters the r.h.s. is due to two reasons. First, now the test function is localized in a neighbourhood of radius $O(\kappa)$ so that κ plays the role normally reserved for d. Second, the formal expansion of G in σ prior to expansion in κ results in a situation with several maximal subgraphs (in the context of this lemma we are dealing only with m-subgraphs). This would normally prevent one from obtaining estimates describing

singular behaviour of subgraphs (recall that in the proofs of [1] one normally deals with one maximal subgraph whose singular plane is the point $p = 0$ so that behaviour near $p = 0$ can be described by dependence on the radius of support of test functions). In the present case, however, the singular product expanded in σ depends parametrically on κ—in an interesting way (here the reader should review the pattern of how the factors G depend on κ and σ—see §9.2). Consider the factors that develop singularities after expansion in σ but prior to expansion in κ. The only dependence on κ that remains in such factors is in their momentum arguments. This means that the eventual expansion in κ will result in an $O(\kappa)$ shift[18] of their singular planes, while after expansion in κ there will remain only one maximal subgraph (G itself) whose singular plane is $p = 0$. Therefore, our standard estimates are still meaningful provided the test functions satisfy the condition in the lemma.

The proof proceds as follows. In section 8 of [1] there was constructed a decomposition of unit isolating the singular planes of maximal (m-) subgraphs of G, whereby a smooth function ϕ_G is assigned to each $\Gamma \in S_{\max}[G]$, so that

$$\sum_{\Gamma\in S_{\max}[G]} \phi_\Gamma(p) \equiv 1. \tag{9.4}$$

It is convenient (in fact, natural) to choose ϕ_Γ to have the form $\phi_\Gamma(p/\kappa)$, so that the decomposition of unit works for all κ. Using (9.4), one can write down the following identity:

$$\langle \mathbf{as}^m_\sigma \circ G\,,\,\varphi\rangle = \sum_{\Gamma\in S_{\max}[G]}\ \sum_{m\le M} \langle \mathbf{as}^m_\sigma \circ \Gamma\,,\,\mathbf{t}^{M-m}_\sigma \circ G\backslash\Gamma\,\phi_\Gamma\,\varphi\rangle.$$

Since every Γ is maximal in itself, the parameter σ appears in its expansion only in combination $\kappa\sigma$. Therefore, the estimate (20.10) from [1] may be applied to $\mathbf{as}^m_\sigma\circ\Gamma$ after replacing $d \to \kappa$ and $\kappa \to \kappa\sigma$:

$$|\langle \mathbf{as}^m_\sigma \circ G\,,\,\varphi\rangle| \le \max_{\Gamma\in S_{\max}[G]}\ \max_{m\le M} (\kappa\sigma)^m \kappa^{-m-\omega_\Gamma} \mathcal{S}\big[\mathbf{t}^{M-m}_\sigma \circ G\backslash\Gamma\,\phi_\Gamma\varphi,\kappa\big]\Lambda(\sigma).$$

(We assume that $\sigma \le 1$ and, hence, $\sigma\kappa \le \kappa$ and $\kappa > \operatorname{rad\,supp}\varphi$.) Noticing that:

$$\big\|\mathbf{t}^{M-m}_\sigma \circ G\backslash\Gamma\big\|^k_{\operatorname{supp}(\phi_\Gamma\varphi)} \le \sigma^{M-m}\kappa^{-d_{G\backslash\Gamma}-k}\Lambda(\sigma,\kappa),$$

we obtain the desired estimate. The second statement of the lemma is proved similarly.

Now let us return to the theorem. We now have to deal with singularities of the expansion in κ and, therefore, with κ-subgraphs. To prove the estimates (ii) and (e) we choose the decomposition of unit $1 = \Phi^\kappa(p) + \Phi_\kappa(p)$ (Φ^κ non-zero in a neighbourhood of $p = 0$), with Φ^κ fixed so that $\forall\Gamma \triangleleft G$ and $\forall g \in G\backslash\Gamma$:

$$\operatorname{supp}(\Phi_\kappa\theta_\Gamma) \cap \mathcal{O}^\kappa_{g(G)} = \emptyset,$$

where $\mathcal{O}^\kappa_{g(G)}$ is the κ-vicinity of g defined as in [1] but with the following modifications. The singular plane π^σ_g of an element g at $\sigma = 0$ can be displaced from the one at $\kappa = 0$, π^κ_g, by const $\times\,\kappa$. Therefore we can take a neighbourhood of the plane π^κ_g with a radius const $\times\,\kappa$ and containing the singular plane π^σ_g for all $\kappa \to 0$.

Since $\operatorname{rad\,supp}\Phi^\kappa = C\kappa$, we use our usual representation of Φ_κ as an integral over spherical layers of radius λ, $\Phi_\kappa = \int_{C\kappa}^d d\lambda/\lambda\,\phi_\lambda$, and get for ($ii$) with φ replaced by $\varphi\phi_\lambda$ in analogy with the proof of (d),(i) for $\lambda > C\kappa$ (using (B.3)):

$$|\langle \mathbf{as}^m_\sigma \circ (1-\mathbf{As}^n_\kappa)\circ G\,,\,\varphi\phi_\lambda\rangle| \le \kappa^{n+1}\sigma^m \sum_{k\ge B_n} \lambda^{k-\omega_G-n-1}\|\varphi\|^k \Lambda(\lambda)\,\Lambda(\kappa,\sigma).$$

Integrating over λ from $C\kappa$ to d we immediately obtain a "half" of (ii) (i.e. for $\varphi\Phi_\kappa$ instead of φ). The second half,

$$\langle \mathbf{as}^m_\sigma \circ (1-\mathbf{As}^n_\kappa)\circ G\,,\,\varphi\Phi^\kappa\rangle = \langle \mathbf{as}^m_\sigma \circ G\,,\,\varphi\Phi^\kappa\rangle - \sum_{l\le m} \langle \mathbf{as}^l_\sigma \circ \mathbf{as}^n_\kappa \circ G\,,\,\varphi\Phi^\kappa\rangle,$$

is estimated with the help of lemma 3 (the first term) and (d) which has been already proved. The estimate (e) may be obtained in the same manner. This completes steps 1–2 of the proof.

[18] and/or rotation in the more general situation of the expansion problem with contact terms.

Step 3. The extension procedure have already been performed for the series $\mathbf{As}_\kappa \circ G$ and $\mathbf{As}_\sigma \circ G$ in [1]. Our purpose is to extend $\mathbf{As}_{\kappa,\sigma} \circ G$ to the distribution on $\mathcal{D}(P)$ while preserving all the estimates. It can be done in two steps. First, one applies the special subtraction operator $\tilde{\mathbf{r}}$ to $\mathbf{as}^{n,m}_{\kappa,\sigma} \circ G$ so that it obey the estimates (d). This procedure is performed similarly to the reasoning of [1]. Second one adjusts the finite renormalization of $\mathbf{as}^{n,m}_{\kappa,\sigma} \circ G$ so that it stisfies (d) (and, of course, (i) and (ii)). Namely, let:

$$\mathbf{as}^{n,m}_{\kappa,\sigma} \circ G = \tilde{\mathbf{r}} \circ \mathbf{as}^{n,m}_{\kappa,\sigma} \circ G + \sum_{|a| = \max(\omega_G + n, 0)} \delta_a(p)\, E^{\kappa,\sigma}_{a,m}. \tag{9.5}$$

It is straightforward to check that the choice

$$E^{\kappa;\sigma}_{a,m} = \langle \mathcal{P}_{a,G} * \left[\mathbf{as}^m_\sigma \circ G - \tilde{\mathbf{r}} \circ {\mathbf{As}'}^{\,n}_\kappa \circ \mathbf{as}^m_\sigma \circ G \right] \rangle \tag{9.6}$$

satisfies all the requirements, which completes the proof of the theorem 1.

A few remarks are in order.

(a) The equations (9.5) and (9.6) give explicit recursive formulae for the double As-expansion. They can be resolved along lines of sect.3 which has already been done in the preceding section.

(b) The theorem can be readily generalized to the case of N-fold expansions. For example, if one wished to study a two-scale expansion of a renormalized diagram, one would have to use a triple As-expansion etc.

(c) In our main inequality (9.3) the graph G is compared to a rather weird expression $\mathbf{As}_\kappa \circ G + \mathbf{As}_\sigma \circ G - \mathbf{As}_{\kappa,\sigma} \circ G$. However, this is only necessary for obtaining factorized bounds. Should one need just an approximation for G irrespective of whether it should be achieved due to smallness of κ or σ or both—as is often the case in applications—it is sufficient to use a suitably truncated series $\mathbf{As}_{\kappa,\sigma} \circ G$.

Conclusions.

A theory that claims to be a comprehensive alternative to the BPHZ method should be able to address at the formal level, as a minimum, the problem of UV renormalization and that of operator product expansions. The theory of As-operation, which has already enjoyed success in applications, has now fulfilled this criterion.

As was observed in [12], [13] the key difference between the two paradigms—BPHZ and our techniques based on the As-operation—is how the basic dilemma of the theory of multiloop Feynman diagrams is resolved. The dilemma consists in the conflict between the inherently recursive nature of the problems of perturbative quantum field theory involving hierarchies of Feynman diagrams, and the singular nature of the objects participating in such recursions. The BPHZ approach consists in systematically getting rid of the singularities by explicitly resolving the corresponding recursive patterns and thus reducing the problem to absolutely convergent integrals. However, those recursive structures are inherently natural, and to ignore them—as the BPHZ approach does—means to lose the heuristic advantage of dealing with complicated objects in a manner respectful of their true nature. Moreover, in the case of non-Euclidean asymptotic regimes, the corresponding recursions are much more complex and can hardly be resolved in an explicit form [13].

The technique of the As-operation, on the contrary, allows one to preserve and make efficient use of the recursive structures by offering means to directly work with singular expressions. The key anlytical idea of the theory of the As-operation is to use the locality condition which explicates the recursive strucures, and thus to reduce the problem to studying the singularities localized at an isolated point. As a result, formal proofs become algebraically explicit and compact, while the final calculational formulae, powerful.

To put it shortly: the BPHZ formalism is only an instrument of formal proof; the techniques of the As-operation is also an instrument of discovery.

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Appendices.

A. Factorizability of the operation $\mathcal{R}$.

Let us prove factorizability of the operation $\mathcal{R}$. It is always possible to fix the associated $\tilde{R}$-operation to be factorized whence the factorization of $\mathcal{R}$ follows. For clarity, we consider the case of two factors.

Let $G'(p',\kappa)$ and $G''(p'',\kappa)$ be two graphs, with p' and p'' independent. We assume that the operation $\mathbf{As}_\kappa$ is well-defined on both of them. We wish to prove that, provided the operation $\tilde{\mathbf{r}}$ is chosen to be factorized, the operation $\mathcal{R} = 1 - \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa$ factorizes as follows:

$$\mathcal{R}\circ\int dp'\, dp''\, G'(p',\kappa)\, G''(p'',\kappa) = \mathcal{R}\circ\int dp'\, G'(p',\kappa) \times \mathcal{R}\circ\int dp''\, G''(p'',\kappa). \tag{A.1}$$

We will present simple arguments which use only factorizability of the operation $\mathbf{As}_\kappa$ (which follows from uniqueness of As-expansions—see §15.4 of [1]) and the fact that the expression $\mathcal{R}\circ\int dp\, G(p)$ (where $p=(p',p'')$) is exactly the coefficient of $\delta(p)$ in the As-expansion of $G(p,\kappa)$ in κ in the sense of distributions:

$$\mathbf{As}_\kappa\circ G(p,\kappa) = \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G(p,\kappa) + \delta(p)\,\mathcal{R}\circ\int d\bar{p}\, G(\bar{p},\kappa) + \text{higher derivatives of } \delta(p). \tag{A.2}$$

The proof runs as follows. First one writes:

$$\begin{aligned}&\mathbf{As}_\kappa\circ\big[G'(p',\kappa)\times G''(p'',\kappa)\big] \\ &= \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ\big[G'(p',\kappa)\times G''(p'',\kappa)\big] + \delta(p')\delta(p'')\,\mathcal{R}\circ\int d\bar{p}' d\bar{p}''\, G'(\bar{p}',\kappa)G''(\bar{p}'',\kappa) + \dots .\end{aligned} \tag{A.3}$$

Then for each factor one has a similar expression; e.g. for G':

$$\mathbf{As}_k\circ G'(p',\kappa) = \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G'(p',\kappa) + \delta(p')\,\mathcal{R}\circ\int d\bar{p}'\, G'(\bar{p}',\kappa) \tag{A.4}$$

Recall that $\mathbf{As}_k$ factorizes (see §15.9 of [1]):

$$\mathbf{As}_\kappa\circ\big[G'(p',\kappa)G''(p'',\kappa)\big] = \mathbf{As}_\kappa\circ G'(p',\kappa)\times\mathbf{As}_\kappa\circ G''(p'',\kappa). \tag{A.5}$$

Substituting (A.3) and (A.4) into (A.5) and comparing terms (taking into account that we always choose $\tilde{\mathbf{r}}$ to be factorizable) one finds, first, that

$$\begin{aligned}&\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ\big[G'(p',\kappa)\times G''(p'',\kappa)\big] = \tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G'(p',\kappa)\times\tilde{\mathbf{r}}\circ\mathbf{As}'_\kappa\circ G''G''(p'',\kappa) \\ &+\mathbf{As}_\kappa\circ\Gamma'(p',\kappa)\times\delta(p'')\,\mathcal{R}\circ\int d\bar{p}''\,\Gamma''(\bar{p}'',\kappa) + \mathbf{As}_\kappa\circ\Gamma''(p'',\kappa)\times\delta(p')\,\mathcal{R}\circ\int d\bar{p}'\,\Gamma'(\bar{p}',\kappa),\end{aligned} \tag{A.6}$$

whence follows (A.1).

It remains to note that the factorization conditions that we always impose on $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{R}}$ ensure factorizability of $\mathcal{R}$.

B. Auxiliary estimates.

Let H be a subproduct of G, $H\subset G$, and let K be a compact region not intersecting any of the singular planes of H, which is formally expressed as follows:

$$K \subset P_{(G)} \setminus \bigcup_{g\in H}\pi_g .$$

Then

$$\big\|\mathbf{t}^n_\kappa\circ\mathbf{t}^m_\sigma\circ H\big\|^k_{p\in\lambda K} < \Big(\frac{\kappa}{\lambda}\Big)^n \sigma^m\,\frac{\Lambda(\lambda,\kappa)}{\lambda^{d_H+k}}, \tag{B.1}$$

$$\left\| \mathbf{t}^n_\kappa \circ (1 - \mathbf{T}^m_\sigma) \circ H \right\|^k_{p \in \lambda K} < \left(\frac{\kappa}{\lambda}\right)^n \sigma^{m+1} \frac{\Lambda(\lambda, \kappa)}{\lambda^{d_H + k}}. \tag{B.2}$$

Moreover, there exists a constant C (depending on H and K) such that for $\lambda > C\kappa$:

$$\left\| (1 - \mathbf{T}^n_\kappa) \circ \mathbf{t}^m_\sigma \circ H \right\|^k_{p \in \lambda K} < \left(\frac{\kappa}{\lambda}\right)^{n+1} \sigma^m \frac{\Lambda(\lambda, \kappa)}{\lambda^{d_H + k}}, \tag{B.3}$$

$$\left\| (1 - \mathbf{T}^n_\kappa) \circ (1 - \mathbf{T}^m_\sigma) \circ H \right\|^k_{p \in \lambda K} < \left(\frac{\kappa}{\lambda}\right)^{n+1} \sigma^{m+1} \frac{\Lambda(\lambda, \kappa)}{\lambda^{d_H + k}}. \tag{B.4}$$

All the above estimates formalize elementary power counting with respect to each of the three parameters—κ, σ and λ.

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Figure Captions.

Fig.1. An example of the *IR*-subgraph is shown with fat lines in (a). One can take p_1 and p_2 for its proper variables p_γ. (b) corresponds to the "projection" $G\backslash\gamma$.

Fig.2. (a) is the graph G, the fat lines constitute one of its *IR*-subgraphs γ. (c) is the diagrammatic representation for γ considered on its own (recall that we consider integrands prior to any loop integrations), while (b) corresponds to $[G\backslash\gamma]_\gamma$.

Fig.3. An example of a general *IR*-subgraph γ (shown with fat lines). Its proper variables are $p_\gamma = (p_1, p_2, p_3)$, while $p_\gamma^{\rm int} = p_2$ and $p_\gamma^{\rm ext} = (p_1, p_3)$. The complement $G\backslash\gamma$ (formed by thin lines) falls into two *UV*-subgraphs ξ_1 and ξ_2 whose loop momenta are $p_{\xi_1} = p_4$ and $p_{\xi_2} = p_5$.

Figures.

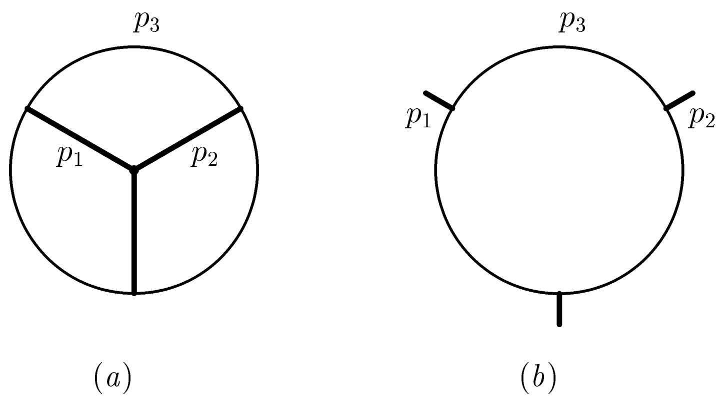

Fig.1

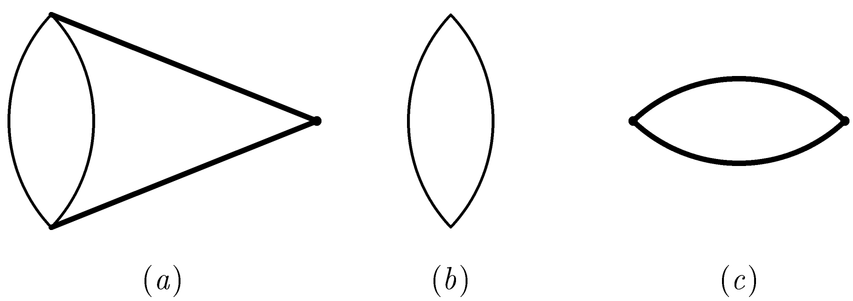

Fig.2

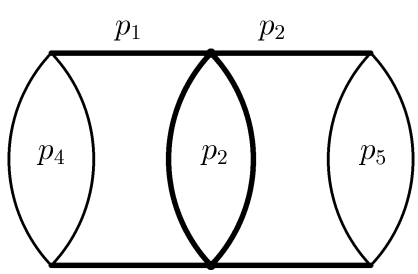

Fig.3